Solution to
Eighth International Mathematics Assessment for Schools
Round 1 of Junior Division

1. What is the simplified value of $(2019-2020)^{2020}+(2018-2019)^{2019}$?

(A) 0 (B) –2019 (C) –1 (D) –2020 (E) 2

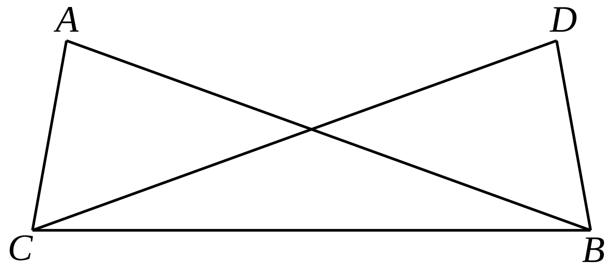
【Suggested Solution】

$$(2019-2020)^{2020}+(2018-2019)^{2019}=(-1)^{2020}+(-1)^{2019}$$
$$=1-1$$
$$=0$$

Hence (A).

Answer：(A)

2. The figure below shows $AB=BC=CD$, $AC=BD$ and $\angle A=80°$. What is the size of $\angle ACD$, in degrees?



(A) 20 (B) 30 (C) 45 (D) 50 (E) 60

【Suggested Solution】

As $AB=BC$, one gets $\angle ACB=\angle A=80°$, then $\angle ABC=180°-80°-80°=20°$.
As $AB=BC=CD$, $AC=BD$, then $\Delta ABC\cong\Delta DCB$, then $\angle DCB=\angle ABC=20°$, so $\angle ACD=\angle ACB-\angle DCB=80°-20°=60°$. Hence (E).

Answer：(E)

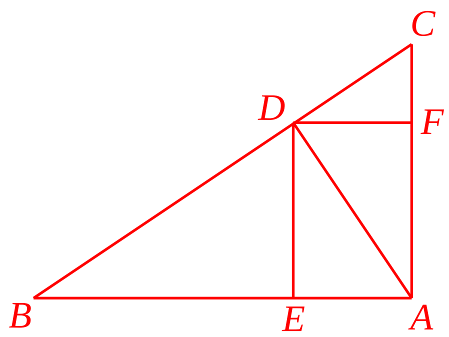
3. In triangle *ABC*, $\angle A$ is the right angle. The line through *A* perpendicular to *BC* intersects *BC* at *D*. Lines through *D* intersect *AB* and *AC* perpendicularly at *E* and *F*, respectively. What is the total number of right triangles in the figure?

(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

【Suggested Solution】

As shown in the figure above, besides the four right triangles *BDE, CDE, ADE, ADF*, there are three more, namely *ABD, ACD, ABC*, with a total 7 in all. Hence (C).

Answer：(C)



4. One third of an acute angle and its complementary angle sums up to $60°$, what is the supplementary angle of this acute angle, in degrees?

(A) 45 (B) 60 (C) 90 (D) 120 (E) 135

【Suggested Solution 1】

Assume this acute angle is $x°$, then $\frac{x}{3}+(90-x)=60$, solve to get $x=45$, so the supplementary angle is $180°-45°=135°$. Hence (D).

【Suggested Solution 2】

The sum of the angle and its complementary angle is $90°$, so 2 third of this angle is $90°-60°=30°$, this angle is $30°\div\frac{2}{3}=45°$, its supplementary angle is $180°-45°=135°$. Hence (D).

Answer：(D)

5. There is an integer n, such that $2020\times(\frac{10}{101})^n$ is an integer. How many possible n's are there?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

【Suggested Solution】

Since $2020=2^2\times5\times101$ and $10=2\times5$, thus the denominator of $(\frac{10}{101})^n$ has to be one of 101, 1 or 10, so n can only be 1, 0 or -1. Hence (C)

Answer：(C)

6. The school restaurant sells three kinds of lunch boxes, the prices of which are \$15, \$13 and \$11, respectively. On a certain day, the total income of the restaurant is \$2019. Which of the following could be the possible number of lunch boxes sold on that day?

(A) 105 (B) 130 (C) 155 (D) 185 (E) 205

【Suggested Solution】

Since each price is an odd number, and the total price 2019 is also odd, then the total number of lunch boxes sold must also be odd. Since the most expensive lunch box is \$15, the number sold must be at least $\frac{2019}{15}=134\frac{4}{15}$; the cheapest box is \$11, thus the number sold is at most $\frac{2019}{11}=183\frac{6}{11}$. Only (C) satisfies. For example, 3 of \$15 boxes, 151 of \$13 boxes and 1 of \$11 boxes gives a total price of $3\times15+151\times13+1\times11=2019$. Hence (C).

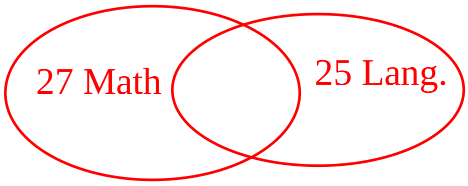
Answer：(C)

7. In an exam, 25 students obtained excellent scores in Language. The number of students with excellent scores in math is 2 more than that in the Language. The students with both excellent scores in Math and Language scores are one third of all students. Every student obtained at least one excellent score. How many students are there in total?

(A) 78 (B) 52 (C) 42 (D) 40 (E) 39

【Suggested Solution】

Suppose there are x students in total. Then there are $25+2=27$ students with excellent math scores. Since every student has some excellent score, we have $25+27=x+\frac{1}{3}x$, that solves to get $x=39$. Hence (E).



Answer：(E)

8. Define the operation $a*b=\frac{a+b+2019}{a-b+2019}$. If $a*2019=7$, what is the value of a?

(A) 673 (B) 674 (C) 1009.5 (D) 1346 (E) 2019

【Suggested Solution】

$7=a*2019=\frac{a+2019+2019}{a-2019+2019}=\frac{a+4038}{a}=1+\frac{4038}{a}$, then $\frac{4038}{a}=6$,

so $a=\frac{4038}{6}=673$. Hence (A).

Answer：(A)

9. Non-negative integer x satisfies $|2x+5|\le 15\times(1+x+x^2+...+x^{2019})^0$. What is the sum of all such x's?

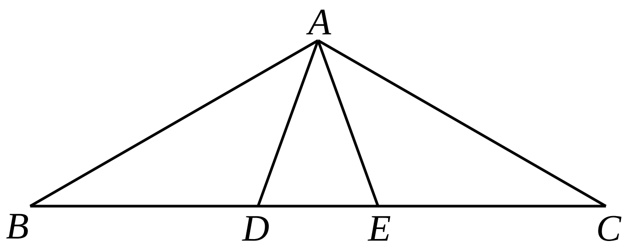
(A) 15 (B) 25 (C) 35 (D) 45 (E) 55

【Suggested Solution】

Since x is non-negative, $|2x+5|=2x+5$; Because $(1+x+x^2+...+x^{2019})^0=1$, so the initial inequality is equivalent to $2x+5\le 15$, so $x\le 5$. The sum is then $0+1+2+3+4+5=15$. Hence (A).

Answer：(A)

10. In a triangle ABC, $AB=AC$. Line segments AD and AE trisect $\angle BAC$, as shown in the figure. If $\angle ADB=110°$, what is $\angle ABC$ in degree?



(A) 30　　(B) 35　　(C) 40　　(D) 45　　(E) 60

【Suggested Solution 1】

$\angle ADE = 180° - 110° = 70°$, by symmetry $\angle AED = 70°$, so $\angle DAE = 180° - 2\times 70° = 40°$, then $\angle BAD = \angle DAE = 40°$. Since $\angle BAD + \angle DAB = \angle ADE$, then $\angle ABC = 70° - 40° = 30°$. Hence (B).

【Suggested Solution 2】

Since $AB = AC$, assume $\angle ABC = \angle ACB = a$, by $\angle BAD + \angle DAB = \angle ADE$, we know $\angle BAD = 70° - a$, that is, $\angle BAD = \angle DAE = \angle EAC = 70° - a$; then by the fact that the sum of angles of a triangle is $180°$, one gets $2a + 3(70° - a) = 180°$, $a = 30°$. Hence (A).

Answer：(A)

11. When $x = 5$, $mx^5 + nx^3 + kx + 2$ is 100. What is the value of $mx^5 + nx^3 + kx + 2$ when $x = -5$?

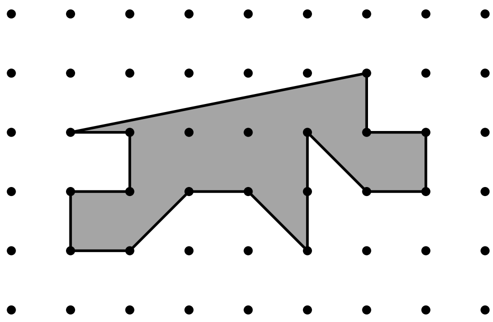
(A) –100　　(B) –99　　(C) –98　　(D) –97　　(E) –96

【Suggested Solution】

Since $5^5m + 5^3n + 5k = 98$, then $(-5)^5m + (-5)^3n + (-5)k = -(5^5m + 5^3n + 5k) = -98$, then answer is $-98 + 2 = -96$. Hence (E).

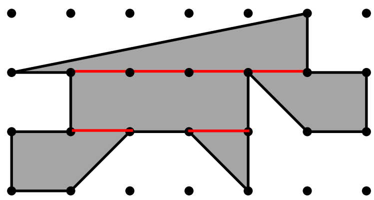
Answer：(E)

12. In the figure below, the distance between two adjacent vertices is 1 cm. What is the area of the shaded portion of the figure, in cm^2?



(A) 7.5　　(B) 8　　(C) 8.5　　(D) 9　　(E) 9.5

【Suggested Solution 1】

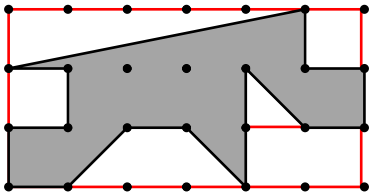


Cut the shaded portion into two right triangles, a rectangle and two trapezoids as shown in the figure, and the area of the shaded part can be computed as

$$\frac{5\times1}{2}+\frac{(1+2)\times1}{2}+\frac{1\times1}{2}+\frac{(1+2)\times1}{2}+1\times3=\frac{5}{2}+\frac{3}{2}+\frac{1}{2}+\frac{3}{2}+3=9\ \text{cm}^2.$$

Therefore, the answer is (D).

【Suggested Solution 2】



As shown in the figure, the area of the shaded part can be known by cutting out from a 3×6 rectangle: two right triangles, two small squares, a rectangle and a trapezoid:

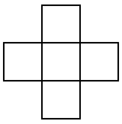
$$3\times6-\frac{5\times1}{2}-1\times1-1\times1-\frac{1\times1}{2}-1\times2-\frac{(1+3)\times1}{2}=18-\frac{5}{2}-1-1-\frac{1}{2}-2-2=9\ \text{cm}^2.$$

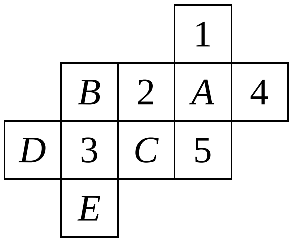
Therefore, the answer is (D).

【Suggested Solution 3】

It can be seen that there are 16 points on side of the shaded region and 2 points inside the shadow region. Therefore, the area of the shaded region can be computed by the Pick Theorem as $2+\frac{16}{2}-1=9\ \text{cm}^2$. Therefore, the answer is (D).

Answer: (D)

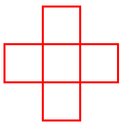
13. Each square in the figure below is filled with a positive integer so that the sum of each  (five squares) is divisible by 3.



What is the minimum value of $A+B+C+D+E$?

(A) 3 (B) 6 (C) 8 (D) 9 (E) 15

【Suggested Solution】

From the give figure, there are two  (of which there are 5 squares each), so

we need to satisfy $1+2+4+5+A$, $3+B+C+D+E$ is divisible by 3.
But $1+2+4+5+A=12+A$, then the minimum value of A is 3.
Since $3+B+C+D+E$ is divisible by 3, then $B+C+D+E$ is also divisible by 3 and we also have $B+C+D+E \geq 1+1+1+1=4$, it follows the minimum value of $B+C+D+E$ is $6=1+1+1+3$.
Hence, the minimum sum of those numbers filled in these five blank squares must be $3+6=9$. Therefore, the answer is (D).

Answer: (D)

14. Ten-digit number $\overline{A20192020B}$ is divisible by 9, where A, B are digits. If the quotient has the ten-thousands digit 0, what is the value of $|A-B|$?

(A) 0　　(B) 1　　(C) 2　　(D) 3　　(E) 4

【Suggested Solution 1】

As the quotient of $\overline{A20192020B}$ divided by 9 has ten thousand digit 0, assume the quotient to $k\times 10^5+\overline{abcd}$, where k is an positive integer and a, b, c, d are digits, so

$$\overline{A20192020B}=9\times(k\times 10^5+\overline{abcd})=9\times k\times 10^5+9\times\overline{abcd},$$

From $\overline{A20192020B}=\overline{A2019}\times 10^5+\overline{2020B}$, one knows both $\overline{A2019}$, $\overline{2020B}$ are multiples of 9. Since $\overline{A2019}$ has sum of digits of $12+A$ and $A\leq 9$, thus only for $A=6$ that $12+A$ is divisible by 9. Similarly, sum of digits of $\overline{2020B}$ is $4+B$, then $B=5$. So $|A-B|=6-5=1$. Hence (B).

【Suggested Solution 2】

Since $\overline{A20192020B}$ is divisible by 9, so is sum of digits $16+A+B$. $A>0$, so $1\leq A+B\leq 18$, then $17\leq 16+A+B\leq 34$, then $16+A+B=18$ or 27.
If $16+A+B=18$, then $A+B=2$. There are two cases:

(i) $A=2$, $B=0$. Then $\overline{A20192020B}\div 9=2201920200\div 9=244657800$, the ten thousand digit is nonzero, not a solution;

(ii) $A=1$, $B=1$. Then $\overline{A20192020B}\div 9=1201920201\div 9=133546689$, not a solution;

If $16+A+B=27$, then $A+B=11$. There are eight cases:

(i) $A=9$, $B=2$. Then $\overline{A20192020B}\div 9=9201920202\div 9=1022435578$, not a solution;

(ii) $A=8$, $B=3$. Then $\overline{A20192020B}\div 9=8201920203\div 9=911324467$, not a solution;

(iii) $A=7$, $B=4$. Then $\overline{A20192020B}\div 9=7201920204\div 9=800213356$, not a solution;

(iv) $A=6$, $B=5$. Then $\overline{A20192020B}\div 9=6201920205\div 9=689102245$, it is a solution, then $|A-B|=6-5=1$;

(v) $A=5$, $B=6$. Then $\overline{A20192020B}\div 9=5201920206\div 9=577991134$, not a solution;

(vi) $A=4$, $B=7$. Then $\overline{A20192020B}\div 9=4201920207\div 9=466880023$, not a solution;

(vii) $A=3$, $B=8$. Then $\overline{A20192020B}\div 9=3201920208\div 9=355768912$, not a solution;

(viii) $A=2$, $B=9$. Then $\overline{A20192020B}\div 9=2201920209\div 9=244657801$, not a solution.

Hence (B).

Answer：(B)

15. Let x be the integers such that $\dfrac{21}{2+x}$ is an integer. How many such x's are there?

(A) 2　　(B) 3　　(C) 4　　(D) 6　　(E) 8

【Suggested Solution】

Since $\dfrac{21}{2+x}$ is integer, thus $2+x$ is a divisor of 21, that is ±1, ±3, ±7, ±21.
There are 8 distinct values of $2+x$, so does x. Hence (E).

Answer：(E)

16. Given that $|a|=3$, $|b|=5$, $|c|=7$, $|d|=11$, what is the least possible value of the expression $ab+ac+ad+bc+bd+cd$?

(A) 0　　(B) –52　　(C) –94　　(D) –100　　(E) –102

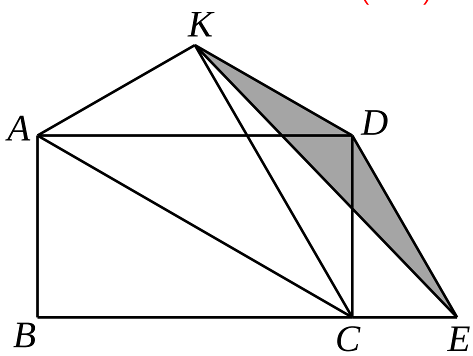
【Suggested Solution】

$$ab+ac+ad+bc+bd+cd=\frac{(a+b+c+d)^2-(a^2+b^2+c^2+d^2)}{2}.$$

Since $a^2+b^2+c^2+d^2=3^2+5^2+7^2+11^2=204$, and a, b, c, d are all odd, $a+b+c+d$ must be a nonzero even number. The least possible value of $(a+b+c+d)^2$ is $2^2=4$, the solution is then $\dfrac{4-204}{2}=-100$, this value happens when $a=3$, $b=-5$, $c=-7$, $d=11$. Hence (D).

Answer：(D)

17. In rectangle $ABCD$, $AB=\sqrt{3}$ cm, $BC=3$ cm. Put a right triangle DCE outside $ABCD$ and along edge CD and $\angle CDE=30^\circ$. Flip along AC such that B lands at K. Join KA, KC, KE, KD, as in the figure. What is the area of the shaded region, in cm^2?



(A) $\frac{3}{2}$　　(B) $\frac{\sqrt{3}}{2}$　　(C) $\frac{\sqrt{3}}{3}$　　(D) $\sqrt{3}-1$　　(E) $\frac{2\sqrt{3}}{3}$

【Suggested Solution】

Since AB: $BC=\sqrt{3}:3=1:\sqrt{3}$ and $\angle ABC=90°$, triangle CAB is a $30°-60°-90°$ right triangle, where $\angle ACB=30°$. Since K is the reflection of B along AC, triangle ACB and ACK are congruent, we get $\angle ACK=\angle ACB=30°$, so $\angle KCD=90°-30°-30°=30°=\angle CDE$, then $KC//DE$, area of KDE and CDE are equal. Since DCE is also a $30°-60°-90°$ triangle, then $CE=\frac{CD}{\sqrt{3}}=\frac{\sqrt{3}}{\sqrt{3}}=1$ cm, thus area of CDE is $\frac{\sqrt{3}\times 1}{2}=\frac{\sqrt{3}}{2}$ cm^2. Hence (B).

Answer：(B)

18. Given that $x=\sqrt{19-8\sqrt{3}}$, which of the following equalities holds true?

(A) $x^2=11$　　(B) $x^2+8x=13$　　(C) $x^2-8x=-13$

(D) $x^2+4x=8$　　(E) $x^2-4x=-8$

【Suggested Solution】

Compute

$x=\sqrt{19-8\sqrt{3}}=\sqrt{19-2\times 4\sqrt{3}}=\sqrt{4^2-2\times 4\sqrt{3}+(\sqrt{3})^2}=\sqrt{(4-\sqrt{3})^2}=4-\sqrt{3}$.

Since $x^2-11=19-8\sqrt{3}-11=8-8\sqrt{3}<0$, (A) is false;

Since $x^2+8x=(19-8\sqrt{3})+8(4-\sqrt{3})=51-16\sqrt{3}\neq 13$, (B) is false;

Since $x^2-8x=(19-8\sqrt{3})-8(4-\sqrt{3})=-13$, (C) is true;

Since $x^2+4x=(19-8\sqrt{3})+4(4-\sqrt{3})=35-12\sqrt{3}\neq 8$, (D) is false;

Since $x^2-4x=(19-8\sqrt{3})-4(4-\sqrt{3})=3-4\sqrt{3}\neq -8$, (E) is false.

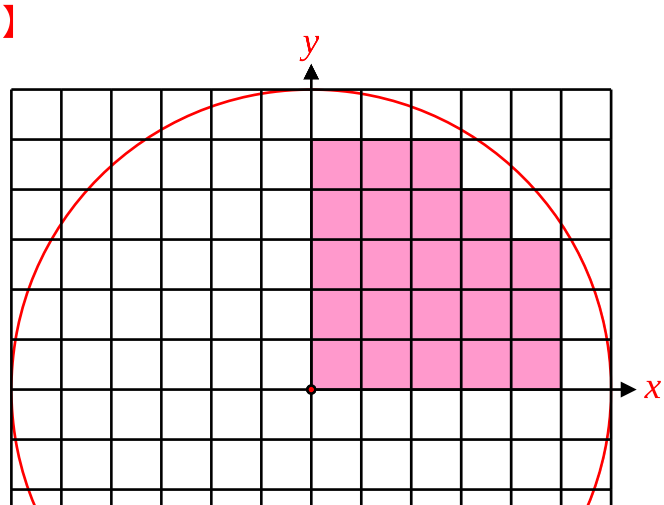
Hence (C).

Answer：(C)

19. Mickey has a large gridded paper. Each small square has side of length 1 cm. He draws a circle with a center at some vertex of a square with radius 6 cm. How many small whole squares are inside the circle?

(A) 64　　(B) 88　　(C) 96　　(D) 100　　(E) 144

【Suggested Solution】



Consider the origin at the center of the circle and *x*, *y* axes along the horizontal or vertical grid lines respectively. Inside the first quadrant, there are 5 squares on the lowest three rows, and 4 squares on the fourth row, 3squares on the fifth. There are 22 complete small squares on the first quadrant, then a total of 88 complete small squares inside the whole circle. Hence (B).

Answer：(B)

20. Suppose $a, b, c, d, e, f, g, h, i, j$ is an arrangement of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 with different letters representing different numbers. At most how many of the ten fractions $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}, \frac{d}{e}, \frac{e}{f}, \frac{f}{g}, \frac{g}{h}, \frac{h}{i}, \frac{i}{j}, \frac{j}{a}$ can be reduced to integer values?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

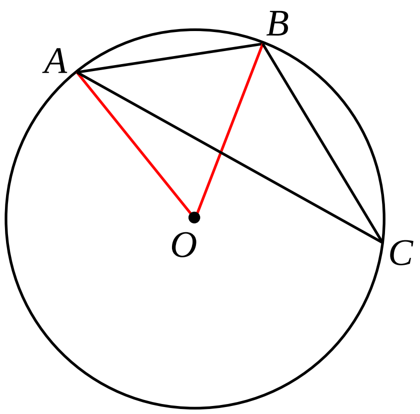
【Suggested Solution】

Since there are no multiples of 6, 7, 8, 9 and 10 in these ten numbers, then the value of a fraction can be an integer only if the denominator does not exceed 5, that is; there are at most 5 integers. On the other hand, desirable $a=8$, $b=4$, $c=2$, $d=1$, $e=6$, $f=3$, $g=10$, $h=5$, $i=9$, $j=7$. It follows

$\frac{a}{b}=\frac{8}{4}=2$, $\frac{b}{c}=\frac{4}{2}=2$, $\frac{c}{d}=\frac{2}{1}=2$, $\frac{e}{f}=\frac{6}{3}=2$, $\frac{g}{h}=\frac{10}{5}=2$ are integers. Hence (B).

Answer：(B)

21. Given *A*, *B*, *C* on circle *O* as shown in the figure, chord $AB=10\,\text{cm}$ and $\angle ACB=30°$. What is the area in cm^2 of circle *O*? (Assume $\pi=3.14$)



【Suggested Solution】

Connect *OA*, *OB*. Then $\angle AOB=2\angle ACB=60°$.
Also $OA=OB$, *OAB* is an equilateral triangle,
thus $OA=OB=AB=10$cm, thus area of circle *O* is
$3.14\times10^2=314\,\text{cm}^2$.

Answer：314

22. Four prime numbers (may not be distinct) *A*, *B*, *C*, *D* satisfy that $(A+1)(A^2+B^2+C^2+D^2)=1149$. What is the value of $A\times B\times C\times D$?

【Suggested Solution】

Since 1149 is an odd number, so $A+1$ and $A^2+B^2+C^2+D^2$ are also odd numbers, it follows A will be an even number, hence $A=2$. Substitute back into the given equality to obtain $B^2+C^2+D^2=\frac{1149}{3}-4=379$. Since every square number has remainder 1 or 0 when divided by 3, and 379 divided by 3 has remainder 1, then it can be concluded that the remainder of two numbers among the three numbers B^2, C^2, D^2 when divided by 3 is 0; that is, two of B, C and D are multiples of 3. Thus, these two numbers must be 3 and the prime number is $\sqrt{379-3^2-3^2}=19$. Therefore, $A\times B\times C\times D=2\times3\times3\times19=342$.

Answer：342

23. Two distinct numbers are selected from 1, 2, 3, …, 100 such that their sum is a factor of 2020. How many different methods of selections are there in total?

【Suggested Solution】

Since $2020=2^2\times5\times101$ and maximum of the sum is 199, minimum is 3. The sum Can only be 4, 5, 10, 20 or 101. Write the chosen numbers as a, b, where $a<b$, then

(i) If $a+b=4$, then (a, b) = (1, 3), 1 choice;

(ii) If $a+b=5$, then (a, b) = (1, 4) or (2, 3), 2 choices;

(iii) If $a+b=10$, then (a, b) = (1, 9), (2, 8), (3, 7) or (4, 6), 4 choices;

(iv) If $a+b=20$, then (a, b) = (1, 19), (2, 18), (3, 17), …, (8, 12) or (9, 11), 9 choices;

(v) If $a+b=101$, then (a, b) = (1, 100), (2, 99), (3, 98), …, (49, 52) or (50, 51), 50 choices.

Totally there are $1+2+4+9+50=66$ number of selections.

Answer：066

24. Integer a and real number x satisfy $x^2-ax+1=0$. What is the minimum value of $\left|x^4+\frac{1}{x^4}-2019\right|$?

【Suggested Solution】

$$x^2-ax+1=0$$

$$x^2+1=ax$$

$$x+\frac{1}{x}=a$$

So $x^2+\frac{1}{x^2}=a^2-2$, $x^4+\frac{1}{x^4}=(a^2-2)^2-2$, so

$$|x^4+\frac{1}{x^4}-2019|=|(a^2-2)^2-2-2019|=|(a^2-2)^2-2021|.$$

Since $44^2=1936<2021<45^2=2025$, when $a^2-2\geq45$, that is, $|a|\geq7$,

$$|(a^2-2)^2-2021|=(a^2-2)^2-2021\ge(7^2-2)^2-2021=188$$

Equality happens when $|a|=7$.

While if $a^2-2\le 44$, that is, $|a|\le 6$,

$$|(a^2-2)^2-2021|=2021-(a^2-2)^2\ge 2021-(6^2-2)^2=865.$$

In summary, the minimum is 188.

Answer：188

25. Arrange the ten numbers 2011, 2012, 2013, …, 2020 randomly on the circumference of a circle, and compute the greatest common divisor of every two adjacent numbers, then add these ten greatest common divisors. What is the maximum of this sum?

【Suggested Solution】

We notice the GCD of any two numbers can't be more than $2020-2011=9$. But except 2016, all the other numbers are not multiples of 6, 7, 8 or 9. So each GCD does not exceed 5.

Since only 2015 and 2020 are multiples of 5, so there is at most one GCD that equals 5. Similarly, there are at most two 4's among the GCD's and at most two 3's, it follows at most four GCD that are even numbers. Thus, the sum can't greater than $5+4+4+3+3+2+2+1+1+1=26$.

In order to obtain the maximum GCD for every two adjacent numbers, it is necessary that 2015, 2020 must be adjacent, then the three numbers 2012, 2016 and 2020 must also be adjacent, then the next three numbers 2013, 2016, 2019 must be adjacent, so there are five even numbers will be adjacent also. At this time, 2020 and 2016 must be at the opposite end of the five even numbers, that is; 2012, 2016, 2020 cannot be adjacent, it is a contradiction. Hence, the sum cannot be more than 25. The arrangement of these 10 numbers are arranged as shown below.

2015

2017 2020

2011 2012

2018 2016

2014 2013

2019

Then the sum is $5+4+4+3+3+1+2+1+1+1=25$.

Answer：025