Solution Key to Second Round of 4th IMAS
Junior Division

1. Which of the following expressions is equal to

$$(a+1)(b-1)+(b+1)(c-1)+(c+1)(a-1)?$$

(A) $ab+bc+ca-3$ (B) $ab+bc+ca$
(C) $ab+bc+ca+2a+2b+2c+3$ (D) $ab+bc+ca-2a-2b-2c-3$
(E) $ab+bc+ca-2a-2b-2c+3$

【Suggested Solution #1】

$$\begin{aligned}&(a+1)(b-1)+(b+1)(c-1)+(c+1)(a-1)\\&=(ab+b-a-1)+(bc+c-b-1)+(ca+a-c-1)\\&=ab+bc+ca-3\end{aligned}$$

【Suggested Solution #2】

When $a=b=c=1$, it follows that the value of the original expression will be 0 and the final results for the expression in each option are:

(A) $ab+bc+ca-3=1+1+1-3=0$
(B) $ab+bc+ca=1+1+1=3$
(C) $ab+bc+ca+2a+2b+2c+3=1+1+1+2+2+2+3=12$
(D) $ab+bc+ca-2a-2b-2c-3=1+1+1-2-2-2-3=-6$
(E) $ab+bc+ca-2a-2b-2c+3=1+1+1-2-2-2+3=0$

Hence, only options (A) and (E) will yield a result of 0.

Now assuming $a=b=c=-1$, so that the value of the original expression will still produce 0 and the results of the expression in each option are as follow:

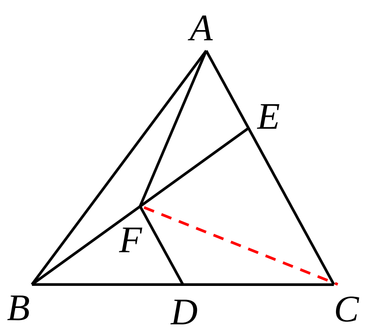
(A) $ab+bc+ca-3=1+1+1-3=0$
(B) $ab+bc+ca=1+1+1=3$
(C) $ab+bc+ca+2a+2b+2c+3=1+1+1-2-2-2+3=0$
(D) $ab+bc+ca-2a-2b-2c-3=1+1+1+2+2+2-3=6$
(E) $ab+bc+ca-2a-2b-2c+3=1+1+1+2+2+2+3=12$

Here, only options (A) and (C) have the same value as that of the original expression, which is 0.

Therefore, only option (A) will produce the same result as the original expression.

Answer: (A)

2. In triangle ABC, D is the midpoint of BC. E is an arbitrary point on CA, and F is the midpoint of BE. If the area of triangle ABC is 120 cm^2 and the area of the quadrilateral $AFDC$ is 80 cm^2, what is the area, in cm^2, of triangle BDF?



(A) 10 cm^2 (B) 15 cm^2 (C) 17.5 cm^2
(D) 20 cm^2 (E) 25 cm^2

【Suggested Solution】

Connect FC. Let S represent the area. From the given information, we know that $S_{\triangle ABF}=S_{\triangle AFE}$, $S_{\triangle BCF}=S_{\triangle FCE}$, which is $S_{\triangle AFC}=\frac{1}{2}S_{\triangle ABC}=60\,\text{cm}^2$. Therefore, $S_{\triangle BDF}=S_{\triangle FDC}=S_{AFDC}-S_{\triangle AFC}=80-60=20\,\text{cm}^2$.

Answer: (D)

3. Let m be a positive integer such that m^3 can be expressed as a sum of m consecutive odd integers. For instance, $2^3=3+5$, $3^3=7+9+11$ and $4^3=13+15+17+19$. If 999 is one of the consecutive odd integers in the expression for m^3, what is the value of m?

(A) 30 (B) 31 (C) 32 (D) 33 (E) 34

【Suggested Solution】

When m^3 was expressed as the sum of m consecutive odd integer and assuming the first term in the series as $2k+1$, then the last term in this series will be represented as $2k+1+2(m-1)=2k+2m-1$, so the sum of the series is $\frac{(2k+1+2k+2m-1)\times m}{2}=m^3$, it follows $2m^3=4k+2m$, then $k=\frac{m^2-m}{2}$, this implies the first term becomes $2k+1=m^2-m+1$ and the last term becomes $2k+2m-1=m^2+m-1$, hence $m^2-m+1\le 999\le m^2+m-1$. Using the completing of square again, then we have

$$m^2-m+1=(m-\frac{1}{2})^2+\frac{3}{4}\le 999\le m^2+m-1=(m+\frac{1}{2})^2-\frac{5}{4}.$$

That is,

$$\begin{cases}(m-\frac{1}{2})^2\le 998\frac{1}{4}<1024=32^2\\ 961=31^2<1000\frac{1}{4}\le(m+\frac{1}{2})^2\end{cases}$$

Thus, we have $m=32$.

Answer: (C)

4. The perimeter of an equilateral triangle is a cm while the perimeter of a square is b cm. If the area of the square is half the area of the triangle, what is the value of $\frac{a^2}{b^2}$?

(A) $\frac{3\sqrt{3}}{8}$ (B) $\frac{3\sqrt{3}}{4}$ (C) $\frac{3\sqrt{3}}{2}$ (D) $\frac{3\sqrt{3}}{3}$ (E) $6\sqrt{3}$

【Suggested Solution】

From the given information, the side length of the given equilateral triangle is $\frac{a}{3}$, by Pythagoras Theorem, the altitude of this equilateral triangle is $\frac{a}{3}\bullet\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{6}a$, then the area of this equilateral triangle is $\frac{1}{2}\bullet\frac{a}{3}\bullet\frac{\sqrt{3}}{6}a=\frac{\sqrt{3}}{36}a^2$. Likewise, the side length of the given square is $\frac{b}{4}$, it follows the area of this square is $\left(\frac{b}{4}\right)^2=\frac{b^2}{16}$, then we have $\frac{b^2}{16}=\frac{1}{2}\bullet\frac{\sqrt{3}}{36}a^2$, this implies $\frac{a^2}{b^2}=\frac{3\sqrt{3}}{2}$.

Answer: （C）

5. Mindy has two boxes, containing 0 and n pieces of candy respectively, where n is a positive integer. She adds 4, 3 and 2 pieces of candy to one of the boxes in that order, always adding to the box containing fewer pieces of candies. If the two boxes have the same number of pieces of candy, then she adds to either of them. In the end, there is 1 more piece of candy in one box than in the other. How many possible values of n are there?

（A）2 （B）3 （C）4 （D）5 （E）6

【Suggested Solution #1】

The usual approach to solve this problem is to generate an equation as follows: $|||n-4|-3|-2|=1$, apply the property of absolute value to the outer of the given absolute value equation, we have $||n-4|-3|-2=1$ or $||n-4|-3|-2=-1$, it follows $||n-4|-3|=3$ or $||n-4|-3|=1$, apply the same property again, then $|n-4|-3=-3, -1, 1, 3$, this implies $|n-4|=0, 2, 4, 6$. Hence $n-4$ will be equal to some of the values in the following: $-6, -4, -2, 0, 2, 4, 6$.

Since n must be positive integers, thus n must be either 2, 4, 6, 8, 10 which is a total of 5 possible values.

【Suggested Solution #2】

Let us solve this problem using Working Backwards, we know that the last two boxes have a total of $n+4+3+2=n+9$ candies and the number of candies in one box is 1 more than the number of candies in the other box, then the total number of candies in the two boxes must be an odd number, it follows n must be an even number. The box that was empty at the start will then have at most $4+3+2=9$ candies at the ending while the box containing n pieces of candies at the start will have at least n pieces of candies at last, then $n-9\le 1$, or $n\le 10$. There are 5 even numbers that are less than 10: 2, 4, 6, 8, 10 and each of them meets the condition of the problem. Hence, the values of n may be 2, 4, 6, 8 and 10. Therefore, there are 5 possible values of n.

Answer: （D）

6. Let a, b and c be real numbers such that x^2+5x-3 is one of the factors of the polynomial x^3+ax^2+bx+c. What is the numerical value of $a+b+2c$?

【Suggested Solution #1】

From the given information, we know that x^3+ax^2+bx+c can be factored as the product of x^2+5x-3 and a linear factor.

Hence, let $x^3+ax^2+bx+c\equiv(x^2+5x-3)(px+q)$, comparing the leading coefficient or x^3 on both sides of the above assumption identity, we have $p=1$, now equating the coefficients of like powers of other terms, we obtain $a=q+5$, $b=5q-3$, $c=-3q$.

Thus, $a+b+2c=(q+5)+(5q-3)+2(-3q)=2$.

【Suggested Solution #2】

We know that x^3+ax^2+bx+c can be expressed as the product of x^2+5x-3 and a linear factor. Since both the coefficient of x^3 and x^2 in x^3+ax^2+bx+c and x^2+5x-3 are equal to 1, it follows the coefficient of x in the linear factor must be equal to 1 as well, this implies the constant in the linear factor is $-\frac{c}{3}$, so that

$$\begin{aligned} x^3+ax^2+bx+c &= (x^2+5x-3)(x-\frac{c}{3}) \\ &= x^3+5x^2-3x-\frac{c}{3}x^2-\frac{5c}{3}x+c \\ &= x^3+(5-\frac{c}{3})x^2-(3+\frac{5c}{3})x+c \end{aligned}$$

then $a=5-\frac{c}{3}$, $b=-\left(3+\frac{5c}{3}\right)$. Thus, $a+b+2c=5-\frac{c}{3}-(3+\frac{5c}{3})+2c=2$.

Answer: 2

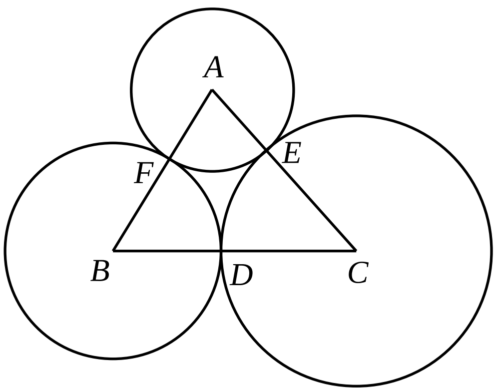
7. How many triples of integers (x, y, z) are such that $|xyz|=6$?

【Suggested Solution】

The given equation can be rewritten as $|x|\times|y|\times|z|=6$; which can be interpreted as the product of three positive integers 6, the possible product is either $1\times1\times6$ or $1\times2\times3$. Consider the three positive integers 1, 1 and 6: there are 3 possible arrangements in the solution of $|x|$, $|y|$, $|z|$. When the three positive integers are 1, 2, 3: there are 6 possible arrangements in the solution of $|x|$, $|y|$, $|z|$. Thus, there are a total of 9 possible arrangements. Lastly, the value of x, y, z may be assigned either positive or negative. Thus, the original equation has a total of $9\times2^3=72$ triples of integer solution.

Answer: 72

8. In triangle ABC, $AB=7$ cm, $AC=8$ cm and $BC=9$ cm. A circle with centre A intersects AB at F and AC at E. The circles with centres B and C and radii BF and CE, respectively, are tangent to each other at a point D on BC. What is the total area, in cm^2, of these three circles? (Taking $\pi=3.14$)



【Suggested Solution】
Let $AF = AE = x\,\text{cm}$,
then $BF = BD = (7-x)\,\text{cm}$, $CE = CD = (8-x)\,\text{cm}$,
so that $(7-x)+(8-x)=9\,\text{cm}$, it follows $x = 3\,\text{cm}$.
This implies the radius of three circles are 3 cm, 4 cm, 5 cm.
Therefore, the area of three circles is $\pi(3^2+4^2+5^2)=50\pi=157\,\text{cm}^2$.

Answer: 157 cm^2

9. There are 2 counters in the first row, and each subsequent row has one more counter than the preceding row. If there are 2015 counters altogether, how many rows of counters are there?

【Suggested Solution】
Assuming all counters are arranged in n rows, then there are $n+1$ in the last row, we obtain the sum of all counters as $\dfrac{n(2+n+1)}{2}=2015$, it follows $n=62$ ($n=-65$ is an extraneous root).,
【Note】 We may use estimation method to solve for $n=62$.

Answer: 62 rows

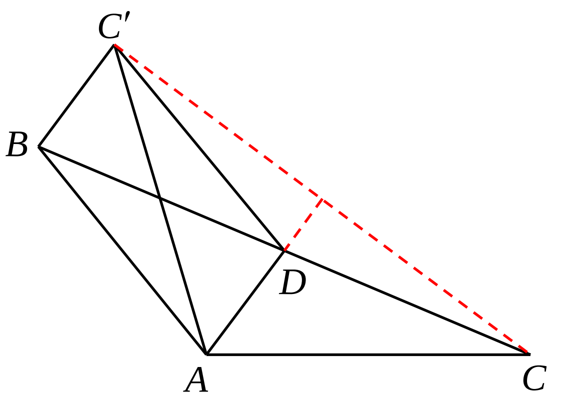
10. In a book fair, the organizers give a book to each participant. Each male participant gives every other male participant a book, and each female participant gives every other female participant a book. If the total number of books received by the male participants is 31 more than the total number of books received by the female participants, how many participants are there altogether?

【Suggested Solution】
Assuming there are m male participants and n female participants attended the book fair. From the given information, we have $m+m(m-1)=n+n(n-1)+31$, rearranging the terms we get $m^2-n^2=31$, then $(m-n)(m+n)=31$. From $m+n>0$ we have $m-n>0$, but 31 is a prime number, then $m-n=1$, $m+n=31$. Therefore, there are 31 participants.

Answer: 31 participants

11. D is a point on the side BC of triangle ABC such that $\angle BAD = 76°$. When the point C is reflected across AD to the point C', $ABC'D$ is a parallelogram. What is the measure, in degrees, of $\angle ADC$?

【Suggested Solution】



Connect CC'. From the given information, $\triangle ADC \cong \triangle ADC'$, then $AC = AC'$ and $\angle CAD = \angle C'AD$. Since the line segments of angle bisector, perpendicular and median of an isosceles triangle coincide together with $AD \perp CC'$, $AD // BC'$,
we conclude that $BC' \perp CC'$.
From the corresponding parts of congruent triangles are equal we have $DC = DC'$, then $\angle DCC' = \angle DC'C$. But $\angle BC'D$ is the complement of $\angle DC'C$ and $\angle C'BD$ is the

complement of $\angle DCC'$, hence $\angle BC'D = \angle C'BD$.
Using Properties of Parallelogram, $\angle ADB = \angle C'BD = \angle BC'D = \angle BAD = 76°$.
Thus, $\angle ADC = 180° - 76° = 104°$.

Answer: 104°

12. Let k be a non-zero integer such that the equation $x + \dfrac{9k^2 - 81}{x} = 10k$ has two distinct integer roots. What is the difference when the smaller root is subtracted from the larger one?

【Suggested Solution】

The original equation can be written as $x^2 - 10k \times x + (9k^2 - 81) = 0$, since both roots are integers, so the discriminant, $\Delta = (10k)^2 - 4(9k^2 - 81)$ must be a perfect square integer. Let $\Delta = m^2$, where m is a positive integer. This implies $m^2 = 64k^2 + 324$, if follows $(m - 8k)(m + 8k) = 324$. ⊗

From the equation ⊗, we know that m is positive integer, then both $m - 8k$ and $m + 8k$ are positive integers, obviously these two integers are distinct due to parity checking. Since $324 = 3^4 \times 2^2$, so we have the following four cases:

Case 1. When $m - 8k = 162$ and $m + 8k = 2$, solving the equation ⊗, we get $m = 82$ and $k = -10$, then the roots of the given quadratic equation are -91 and -9, meet the condition of the problem, so the difference is 82.

Case 2. When $m - 8k = 54$ and $m + 8k = 6$, solving the equation ⊗, we get $m = 30$ and $k = -3$, then the solutions of the given quadratic equation are -30 and 0, but 0 is an extraneous root, which contradicts the given condition, reject the answer.

Case 3. When4 $m - 8k = 6$ and $m + 8k = 54$, solving the equation ⊗, again, this is same with Case 2, an extraneous root appears, this is a contradiction of the given, reject the answer.

Case 4. When $m - 8k = 2$ and $m + 8k = 162$, solving the equation ⊗, it is the same as Case 1, the roots of the given quadratic equation are 9 and 91, meeting the condition of the problem, the difference is 82.

From the above four cases, we conclude the difference of two roots is 82.

Answer: 82

13. The integers 1, 2, 3, …, 20 are divided into two groups. The sum of all the numbers in one group is n, while the product of all the numbers in the other group is also n. What is the maximum value of n?

【Suggested Solution #1】

Let us show that when $n = 192$, it meets the condition of the problem. Let 2, 4 and 8 as the numbers in the second group while all the remaining numbers are in the first group, so the product of the three numbers in the second group is $4 \times 6 \times 8 = 192$ and the sum of all the remaining numbers in the first group is
$(1 + 2 + \cdots + 20) - (4 + 6 + 8) = 192$, which meets the condition of the problem.
None of $n > 192$ will meet the requirements of the problem.

We know the prime factor of each $n=193$, 194, 197, 199, 201, 202 and 203 is greater than 20 and they are 193, 97, 197, 199, 67, 101 and 29; respectively, which cannot be the product of the second group.

When $n=195$, one of the prime factors of n is 13, the second group must contain the number 13, so the sum of all numbers after removing from the second group will be $(1+2+\cdots+20)-195-13=2$, then the second group must contain numbers 2 and 3, but $2\times13\neq195$, a contradiction!

When $n=196$, two of the prime factors of n are 7, the second group must contain 7 and 14, but the sum of all the numbers in the first group is at most $(1+2+\cdots+20)-(7+14)=189$, again a contradiction!

When $n=198$, one of the prime factors of n will be 11, then 11 must be included in the second group, but the sum of all numbers after removing from the second group will be $(1+2+\cdots+20)-198-11=1$, that is, 1 and 11 will be the two numbers in the second group, but $1\times11\neq198$, a contradiction!

When $n=200$, two of the prime factors of n are 5, so there are two numbers in second group that are multiple of 5, the sum of all the numbers in the first group is at most $(1+2+\cdots+20)-(5+10)=195$, a contradiction!

When $n\geq204$, sum of all the numbers in the second group cannot be more than $(1+2+\cdots+20)-204=6$, their product is obviously not equal to n, which is also a contradiction.

In summary, the maximum value of n is 192.

【Suggested Solution #2】

Since $4\times6\times8=192=1+2+3+5+7+9+10+11+\cdots+20$, then the largest possible value of n is at least 192.

There is another case: $1\times2\times3\times4\times8=192=5+6+7+9+10+11+\cdots+20$.

If $n>192$, then the number n must be expressed as the product of m positive integers such that the sum of these m distinct positive integers is at most 17.

From $1+2+3+4+5+6=21$, we know that $m<6$.

When $m=5$, we have $n\leq6\times5\times3\times2\times1=180$, a contradiction!

When $m=3$, we have $n\leq7\times6\times4=168$, a contradiction!

When $m=2$, we have $n\leq8\times9=72$, also a contradiction!

When $m=4$, suppose one of the addends is in 17 is 1 with the largest possible product $n\leq1\times7\times5\times4=140$, a contradiction!

The expressions are $8+4+3+2$, $7+5+3+2$ and $6+5+4+2$ with 192, 210 and 240 as their respective product. From those three products, only 210 and 240 are greater than 192.

But $n<1+2+3+\cdots+20=210$, then the product of these four positive integers cannot be equal to 210 or 240.

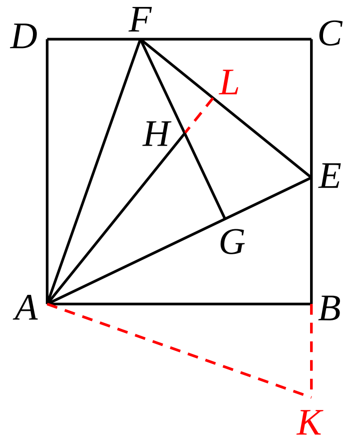
Now consider this case, when the sum of four positive integers is 16, then the largest possible product is $n\leq6\times5\times3\times2=180$, which is also a contradiction!

In summary, the largest required number n is 192.

Answer: 192

14. E is a point on the side BC and F is a point on the side CD of a square $ABCD$ such that the perimeter of triangle CEF is equal to half the perimeter of $ABCD$. G is the point on AE such that FG is perpendicular to AE, and H is the point on FG such that $AH = EF$. Prove that AH is perpendicular to EF.

【Suggested Solution】



Refer to the diagram, extend CB to K such that $BK = DF$ and connect AK. Connect AH to meet EF at L. Since $AB = AD$, $\angle ABK = \angle D = 90°$ so that $\triangle ADF \cong \triangle ABK$. Hence, $DF = BK$, $AF = AK$, $\angle BAK = \angle DAF$. It follows that

$\angle FAK = \angle FAB + \angle BAK = \angle FAB + \angle DAF = \angle DAB = 90°$.

But the perimeter of $\triangle CEF$ is one-half the perimeter of square $ABCD$,

then $EF = 2AB - (CE + CF) = BE + DF = BE + BK = EK$.

But $AE = AE$, $AF = AK$, then $\triangle AFE \cong \triangle AKE$.

Thus, $\angle FAE = \angle EAK = \dfrac{1}{2}\angle FAK = 45°$.

We know that $FG \perp AE$, then $\triangle AGE$ is an isosceles right-angled triangle, so that $AG = FG$.

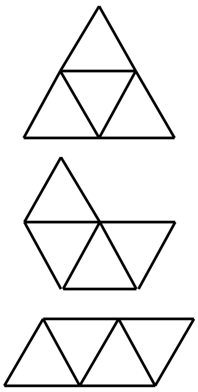
Hence, $AH = EF$, $\angle FAK = \angle AGH = \angle FGE = 90°$ and then $\triangle AGH \cong \triangle FGE$. So that $\angle FLH = 180° - \angle FHL - \angle HFL = 180° - \angle AHG - \angle GAH = 90°$.

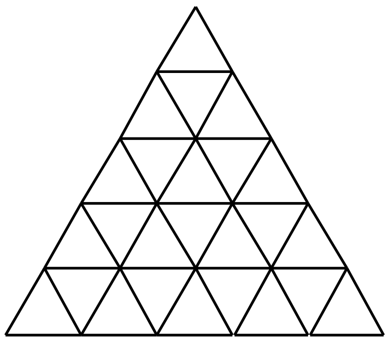
Therefore, $AH \perp EF$.

【Marking Scheme】

- Able to show that $EF = EK$, reward 5 marks.
- Able to show that $\angle FAE = 45°$, reward 5 marks.
- Able to show that $AG = FG$, reward 5 marks.
- Able to show that $AH \perp EF$, reward 5 marks.

15. Each side of an equilateral triangle is divided into 5 equal parts by 4 points, and these points are joined by lines parallel to the sides of the triangle, dividing into 25 small equilateral triangles. A tetriamond is a shape formed from 4 small equilateral triangles joined side to side. There are three tetriamonds as shown in the diagram below on the left.



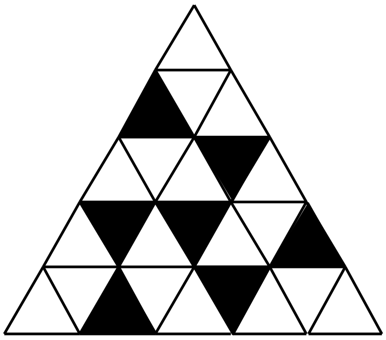


(a) Show that if 7 of the small triangles are painted, then it will be impossible to fit any tetriamond inside the large triangle without covering any part of the painted small triangles. (4 marks)

(b) Prove that if 6 of the small triangles are painted, then it is always possible to fit a tetriamond inside the large triangle without covering any part of the painted small triangles. (16 marks)

【Suggested Solution】

(a) Let us paint 7 of the small triangles black as shown in the figure below, then we cannot put any of the tetriamond in the big triangle by preventing covering any of black-shaded small triangles. When we flip and rotate the biggest triangle, and still is unsuccessful to put any of the tetriamond in the big triangle.

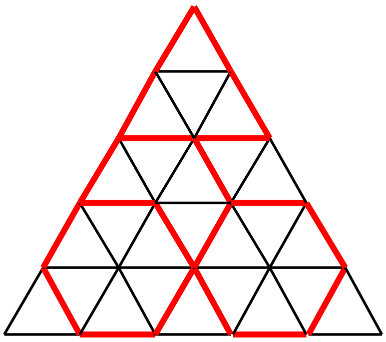


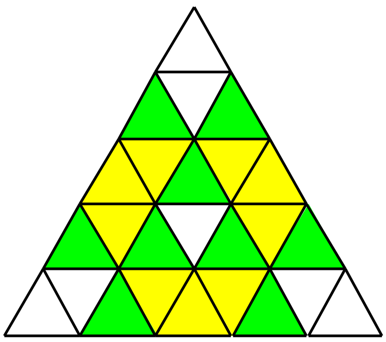
(b) Assuming the conclusion does not hold, then there exists a certain way to color the six of the small equilateral triangles such that it will be impossible to put the tetriamond inside the biggest triangle and will cover all the small equilateral triangle that were painted.

Refer to the figure at the lower left, in the two hexagons whose sides are bounded by red color, there will be at least two small equilateral triangles that must be colored, otherwise it will be easy to locate those tetriamonds that were inside them.

For those small equilateral triangle that were located inside the triangle whose sides were bounded by red, it is necessary to have at least a small equilateral triangle to be colored, for the concave hexagon whose sides are bounded red must also have at least one small equilateral triangle to be colored.

It means the all the 6 small equilateral triangles that will be colored must be inside the triangle whose sides were bounded by red, this implies the two small triangles located on the left side at the center cannot be colored, or else the number that will be colored will exceed 6 pieces.





The cardboard flip rotation, by the symmetry of the figure to the right shows a small

yellow equilateral triangle that is not to be painted, so the little green equilateral triangle must have been painted, otherwise we can find four pieces of equilateral triangle, but there are 9 small green equilateral triangles, which is a contradiction!
In summary, the selection of any 6 small equilateral triangles and color them, they must be able to place in a block of four equilateral triangles and will not cover any of those painted small equilateral triangles in the large equilateral triangle. QED.

【Marking Scheme】

(1) In the first sub-problem: if the method in painting black to the respective small triangles are correct, then reward 4 marks.

(2) For sub-problem:

- Able to point out at least 2 small equilateral triangles in each of the two convex hexagons whose sides are bounded by red must have dig and removed, then reward 4 marks.
- Able to point out that at least one of those small equilateral triangles that are inside those concave hexagons whose sides are bounded by red must dig and removed, then reward 4 marks.
- Able to point out that at least one of those small equilateral triangles that are inside those triangles bounded by red color must be dig and removed, then rewards 4 marks.
- Able to point out that all the small equilateral triangles painted green must be dig and removed, the reward 4 marks.