Solution Key to Second Round of IMAS 2018/2019
Junior Division

1. Which of the following statement is false?
（A） If a divides b and k is an integer, then a divides kb.
（B） If a divides b and b divides c, then a divides c.
（C） If $a = bc$, and b, c are positive integers, then a is divisible by b or c.
（D） If b divides a and c divides a, then bc divides a.
（E） If $p|bc$, then $p|b$ or $p|c$, where p is a prime number, b and c are integers.

【Solution】
Counterexample: 4 divides 12 and 6 divides 12, but $4\times 6 = 24$ does not divide 12. If b divides a and c divides a, then bc does not always divide a, hence (D) is the correct answer. hence (D).

Answer：（D）

2. How many positive integers from 1 to 2019 can be expressed as $n^3 - 3n^2 + 2n$, where n is an positive integer?
（A） 11 （B） 12 （C） 13 （D） 44 （E） 45

【Solution】
$n^3 - 3n^2 + 2n = n(n-1)(n-2)$, which is a non-decreasing function of n. Since $11\times 12\times 13 < 2019 < 12\times 13\times 14$, and $1 \le 3\times 2\times 1$. n can take the values of 3, 4, …, 13 and in total, there are 11 different values, hence (A).

Answer：（A）

3. The perimeter of an isosceles triangle is known to be 32 cm and length of each side is an integer, in cm. How many different non-identical such triangles are there?
（A） 5 （B） 6 （C） 7 （D） 8 （E） 9

【Solution】
Assume the isosceles side has length x cm, then the bottom side has length $(32-2x)$ cm. Since $0 < 32-2x < 2x$, thus $8 < x < 16$. There are 7 choices of x as 9, 10, 11, 12, 13, 14, 15, hence (C).

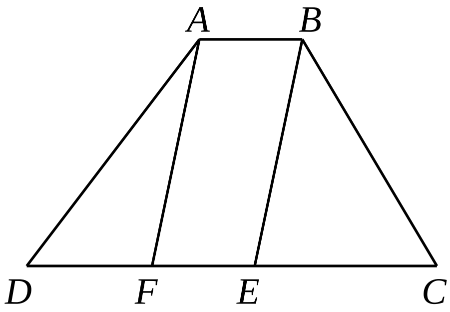
Answer：（C）

4. Given four distinct non-zero digits a, b, c and d, if $\overline{ab} + \overline{cd} = \overline{dc} + \overline{ba}$, then this expression is called a palindrome expression and the sum of the two numbers $\overline{ab} + \overline{cd}$ is called a palindrome sum. For example, $53 + 46 = 64 + 35 = 99$. What is the minimum possible value of a palindrome sum?
（A） 22 （B） 33 （C） 44 （D） 55 （E） 99

【Solution】
Since $\overline{ab} + \overline{cd} = \overline{dc} + \overline{ba}$, it is known that $10(a+c) + (b+d) = 10(b+d) + (a+c)$, thus $a + c = b + d$. The least number that can be represented as the sum of two different positive numbers in two different ways is $5 = 1+4 = 2+3$, thus the least palindrome sum is 55, for example, $12 + 43 = 34 + 21 = 55$. Hence (D).

Answer：（D）

5. In the figure below, the area of the trapezoid $ABCD$ is 100 cm^2, the area of parallelogram $ABEF$ is 40 cm^2 and $CD=10$ cm. What is the length, in cm, of AB?



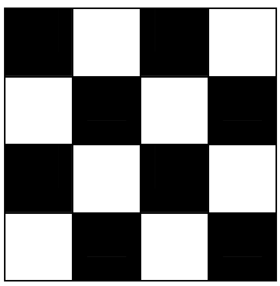
（A）2 （B）2.5 （C）4 （D）5 （E）10

【Solution】

If $AB=a$ cm, and height of the trapezoid is h cm, we have $ah=40$ cm^2, $\frac{(a+10)h}{2}=100$ cm^2, then $a+10=5a$, $a=2.5$, hence (B).

Answer：(B)

6. Four identical chess pieces are to be placed into a 4×4 chess board that is colored black and white alternately, as shown in the figure below. You can place at most one chess piece on each square. All chess pieces must be placed in squares of the same color and no two pieces are on the same row or on the same column. In how many different ways can the chess pieces be placed?



【Solution】

If all pieces were placed into black squares, there are two ways to put the pieces on the first and the third row, two ways to put the pieces on the second and the fourth row, totally there are 4 ways. Similarly, there are 4 ways to place all pieces into white squares. Hence there are 8 ways in total.

Answer：8 ways

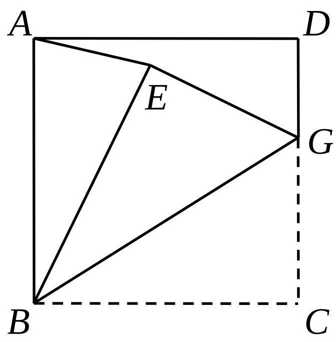
7. It is known that for any $x\neq\pm\frac{1}{2}$, then $\frac{a}{x+\frac{1}{2}}+\frac{b}{x-\frac{1}{2}}=\frac{24x+4}{4x^2-1}$. What is the value of $a+b$?

【Solution】

Multiply both sides by $4x^2-1$ to obtain $2a(2x-1)+2b(2x+1)=24x+4$, comparing coefficients of x, $4a+4b=24$, thus $a+b=6$.

Answer： 6

8. In the figure below, $ABCD$ is a square and the point G lies on the side CD. Now, flip triangle BCG along BG to get a new triangle BEG. If $\angle CBG = 32°$, then what is the size, in degrees, of $\angle DAE$?



【Solution】

Since $\angle ABE = 90° - 2\angle CBG = 26°$. Since $AB = BC = BE$, triangle BEA is an isosceles triangle, we have $\angle BAE = \dfrac{180° - 26°}{2} = 77°$, and $\angle DAE = 90° - 77° = 13°$.

Answer： 13°

9. How many ordered triples (a, b, c) of integers that satisfy the equation $|ab| + |bc| + |ca| = 9$?

【Solution】

Let's consider a case where at least one of a, b, c is 0. If $a = 0$, then $bc = \pm 9$, then b can take ± 1, ± 3, ± 9, and after b is taken, c has two possible values. There are 12 such solutions. Similarly, there are 12 solutions when $b = 0$ or $c = 0$. If a, b, c are all nonzero, then one of $|ab|$, $|bc|$, $|ca|$ is no more than 3 and hence at least one of a, b, c has an absolute value of 1. Assume $|a| = 1$, then $(|b|+1)(|c|+1) = 10$, one of $|b|$, $|c|$ is 1 and another is 4. Thus for non-zero solutions, one of the three numbers has absolute value 4, the other two has absolute value 1. There are $2^3 = 8$ such solutions. In total, there are $12\times 3 + 8\times 3 = 60$ triples.

Answer：60 triples

10. If $x + y = \sqrt{4z-1}$, $y + z = \sqrt{4x-1}$ and $z + x = \sqrt{4y-1}$, where x, y and z are real numbers, then what is value of $x + y + z$?

【Solution】

Add the three equations to get $2x + 2y + 2z = \sqrt{4x-1} + \sqrt{4y-1} + \sqrt{4z-1}$.

Combining terms, we have,

$$2x + 2y + 2z - \sqrt{4x-1} - \sqrt{4y-1} - \sqrt{4z-1} = 0 \qquad (1)$$

Now, divide both sides of (1) by 2,

$$x + y + z - \sqrt{x - \frac{1}{4}} - \sqrt{y - \frac{1}{4}} - \sqrt{z - \frac{1}{4}} = 0 \qquad (2)$$

As $x - \frac{1}{4} - \sqrt{x - \frac{1}{4}} + \frac{1}{4} = (\sqrt{x - \frac{1}{4}} - \frac{1}{2})^2$, (2) can be written as

$$(\sqrt{x - \frac{1}{4}} - \frac{1}{2})^2 + (\sqrt{y - \frac{1}{4}} - \frac{1}{2})^2 + (\sqrt{z - \frac{1}{4}} - \frac{1}{2})^2 = 0$$

Then $\sqrt{x - \frac{1}{4}} - \frac{1}{2} = \sqrt{y - \frac{1}{4}} - \frac{1}{2} = \sqrt{z - \frac{1}{4}} - \frac{1}{2} = 0$, $x = y = z = \frac{1}{2}$, and $x + y + z = \frac{3}{2}$.

Answer： $\dfrac{3}{2}$

11. Place 9 distinct positive integers into each of the unit squares of the 3×3 square below, with one number in each unit square, such that the sum of the numbers in every 2×2 square is 50. What is the minimum possible value of sum of the 9 integers?

【Solution】

Represent the numbers of each unit square by one letter as in the figure below.

a	b	c
d	e	f
g	h	i

Now, compute the sum of numbers of each 2×2 square:

$$a+b+d+e=50$$

$$d+e+g+h=50$$

$$b+c+e+f=50$$

$$e+f+h+i=50$$

Add the four equations to get $(a+c+g+i)+2(b+d+f+h)+4e=200$.

If sum of the nine numbers is S, then

$$\begin{aligned}4S&=4(a+c+g+i)+4(b+d+f+h)+4e\\&=200+3(a+c+g+i)+2(b+d+f+h)\\&=200+(a+c+g+i)+2(a+b+c+d+f+g+h+i)\end{aligned}$$

But

$$a+b+c+d+f+g+h+i\geq1+2+3+4+5+6+7+8=36$$

$$a+c+g+i\geq1+2+3+4=10$$

So $4S\geq200+10+2\times36=282$, $S\geq70.5$. As S is an integer, $S\geq71$.

On the other hand, the following construction satisfies that $S=71$.

1	9	2
6	34	5
3	7	4

Answer：71

12. The three side lengths of an acute triangle are consecutive integers, in cm, and it is known that one altitude on one side is 12 cm. What is the area, in cm^2, of this triangle?

【Solution】

Assume the length of other two sides are *a*, *b*, by Pythagorean theorem, the altitude of length 12 cm is on a side of length $\sqrt{a^2-12^2}+\sqrt{b^2-12^2}$. The two numbers with square root sign have to be integers. Consider positive integer solutions to $x^2-12^2=y^2$ via $(x+y)(x-y)=12^2=2^4\times 3^2$, one finds out $x=13$, 15, 20, 37. Only 13, 15 are possibly two of three consecutive integers. Check that the third side length is $\sqrt{13^2-12^2}+\sqrt{15^2-12^2}=5+9=14$ cm. Thus, the area of this triangle is $\frac{14\times 12}{2}=84$ cm^2.

Answer： 84 cm^2

13. Arrange all positive integers less than 30 and not divisible by 3 in an increasing order, and compute the sum of the reciprocals of product of every three consecutive numbers, that is $S=\frac{1}{1\times 2\times 4}+\frac{1}{2\times 4\times 5}+\cdots+\frac{1}{26\times 28\times 29}$. Now, if we reduce S into its simplest form, then what would be the value of the numerator?

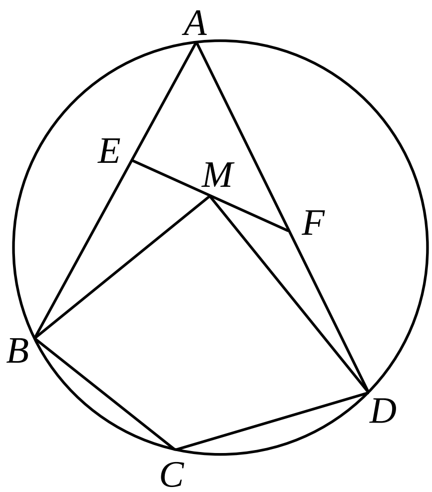
【Solution】

$$\begin{aligned}3S &= \frac{3}{1\times 2\times 4}+\frac{3}{2\times 4\times 5}+\cdots+\frac{3}{26\times 28\times 29}\\ &=\left(\frac{1}{1\times 2}-\frac{1}{2\times 4}\right)+\left(\frac{1}{2\times 4}-\frac{1}{4\times 5}\right)+\cdots+\left(\frac{1}{26\times 28}-\frac{1}{28\times 29}\right)\\ &=\frac{1}{2}-\frac{1}{28\times 29}\\ &=\frac{14\times 29-1}{28\times 29}=\frac{405}{812}\end{aligned}$$

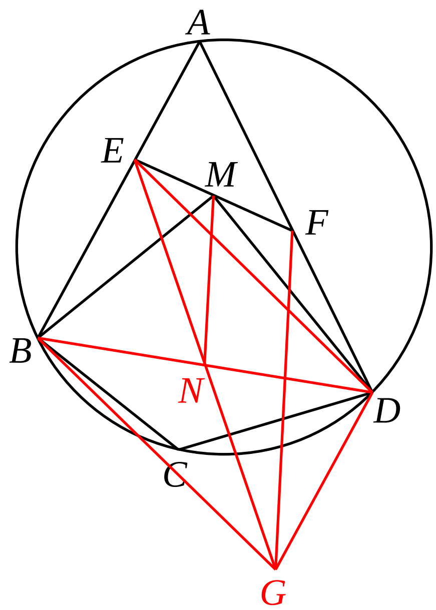
Thus $S=\frac{135}{812}$.

Answer： 135

14. In the figure below, a convex quadrilateral *ABCD* is inscribed in circle *O*. Points *E* and *F* are on segments *AB* and *AD* respectively such that $BE=CD$ and $DF=BC$. If point *M* is the midpoint of *EF*, then prove that $BM\perp DM$.



【Solution】



As shown in the figure, let the midpoint of BD be N, extend EN to G, such that $EN = NG$. (5 points) Connect ED, DG, FG, BG, then $BEDG$ is a parallelogram. Then $DG = BE = CD$, and $\angle GDF = 180° - \angle A = \angle DCB$, $DF = BC$, hence $\triangle GDF \cong \triangle DCB$, and $FG = BD$ （5 points）. As MN is median line of triangle EFG, $MN = \frac{1}{2}FG$ （5 points）. So $MN = \frac{1}{2}BD = BN = ND$, then $BM \perp DM$ （5 points）.

15. A robot can generate a set of digit codes according to user's reasonable instructions. Wayne gives out the following commands:

(1) Each code is a four-digit number (nonzero for the left-most digit).

(2) Every two codes in the set have identical digits at no more than two corresponding positions.

Find the maximum number of codes in a set the robot can generate.

【Solution 1】

The left most digit contains 9 possible digits. If the total number of codes is more than 900, then by the pigeon cage principle, at least two codes have identical first three digits, a contradiction. (5 points)

Construct 900 codes as follows: the first three digits take 100 to 999, once for each; the fourth digit equals the last digit of sum of first three digits. (5 points)

Next, we show that such set of codes satisfies the command. For any code $\overline{abcd}$, $a+b+c-d$ is divisible by 10, thus each of a, b, c, d is determined by the other three. If two codes have identical digits at three corresponding places, their digits at the fourth place are also the same, showing that they are the same code. Thus every two codes have identical digits at no more than two places. (10 points)

【Solution 2】

List 900 codes satisfying the command. (10 points, any error or missing results in 0 points)

Prove that there are at most 900 codes. (10 points)

Answer：900 codes

【Note】

We can construct 900 codes as follows: the first three digits take 100 to 999, once for each; the forth digit satisfies that the sum of four digits is a multiple of 10. If two codes have two identical digits and the other one is different in the corresponding position, then the forth digits are also different.