2011Junior form (Secondary)
Solution for Second-round test

1. The order of digits of the reflected image formed under water does not change, but the digits are inverted. Here, 0 and 1 do not change but 2 is changed to 5. So the reflected image of the number formed under water shows 5015. Hence B.

Answer：(B)

2. Assume there are n teams, i.e. $\frac{3n}{2}+\frac{(n-6)(n-7)}{2}=33$, then $n=12$ or $n=-2$ (rejected). Hence D.

Answer：(D)

3. 【Solution 1】

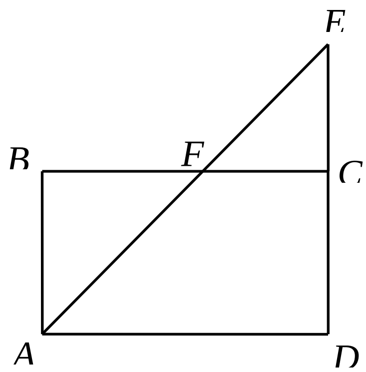
Let the sum of age of the four children be x. Then the age of the four children can be represented as $x-22$, $x-20$, $x-17$ and $x-25$ respectively. Then $(x-22)+(x-20)+(x-17)+(x-25)=x$ and x = 28. Hence the age of these four children are 6, 8, 11, and 3 years respectively. The eldest one is 11 years old and the youngest is 3 years old. Hence the difference between the eldest and the youngest is 8 years.

【Solution 2】

Let the sum of age of the four children be x. Then the age of the four children can be represented as $x-22$, $x-20$, $x-17$ and $x-25$ respectively. Given that the eldest one is $x-17$ years old and the youngest one is $x-25$ years old, the difference between the eldest and the youngest is $(x-17)-(x-25)$ = 8 years. Hence E.

Answer：(E)

4. Let the distance travelled by three ants A, B and C be x, y and z respectively. Firstly, compare x and z, as $x=AB+BC=CD+BF+CF$, $z=AF+FC+CD$, so we only need to compare the two lengths AF and BF. Since the hypotenuse is longer than the leg, so $AF>BF$, therefore $x<z$. Then we compare y and z. As $y=AF+EF+EC+CD$, so we only need to compare the two lengths CF and $EF+EC$. As the shortest distance between two points is a straight line, so $CF<EF+EC$, which implies $z<y$. As a result, $x<z<y$. Since the time used to reach the destination is proportional to the length of the route, the order for three ants to reach the destination (from the earliest to the latest) is A, B, C. Hence B.



Answer：(B)

5. If the price of a box of caramel is two dollars, the two customers need to pay 4 dollars altogether to Andy. This must be paid by four one dollar bills. Hence one of the customers should at least have two one dollar bills. If this is the case, this

customer could pay his caramel, this is a contradiction.

If the price of a box of caramel is five dollars, the total amount paid is 10 dollars. This could be 10 one-dollar bill, 5 one-dollar bills and 1 five-dollar bill or 2 five-dollar bill. In any circumstances, one of the customers must have a five-dollar bill, and he can pay his caramel, this is a contradiction.

If the price of a box of caramel is six dollars, the total amount paid is 12 dollars. For twelve dollars, two must be one-dollar bills, so the one who has a one dollar bill could not have a five-dollar bill, otherwise he can pay his caramel. Therefore, he only has one-dollar bills (the number of one-dollar bills cannot exceed five) and ten-dollar bills (bills larger than 10 dollars is not considered, as it could be exchanged to ten dollars bill). If he has eleven dollars, he should get back five dollars from the other customer. So no matter he gives the other a one-dollar bill or ten-dollar bill, the other customer could paid his fare originally. Similarly, it proves that it covered the case of 12, 13, 14, 15 dollars. And the price of a box of caramel cannot be six dollars.

If the price of a box of caramel is seven dollars, the total amount paid is 14 dollars. For 14 dollars, four must be one-dollar bills. This implies that one of the customers has at least 2 one-dollar bills. Then he should not have a five-dollar bill, otherwise he can pay his fare. The same as the above argument, the price of a box of caramel cannot be seven dollars.

Suppose the price of a box of caramel is eight dollars. If one of the customer has a ten-dollar bill and 3 one-dollar bills and the other person has 2 five-dollar bills (the price of a box caramel is eight dollars), the first costumer can give 2 one-dollar bill to the second one, and the second one give a five-dollar bill to the first one. Hence E.

Answer：(E)

6. Let the length of zig-zag line AC be x and the speed of the Peter be v, then the speed of the Toney is $\frac{3}{4}v$. From the question, we have $\frac{x+2}{v}=\frac{x-2}{\frac{3}{4}v}$, which gives us the result $x=14$ km.

Answer：14 km

7. 【Solution 1】
If the first digit on the key is 1, the corresponding room could be 1, 4, 7, 10, 13, 16 and19.
If the second digit on the key is 2, the corresponding room can be 2, 9 and16.
So if a two-digit number 12 is printed on the key, the corresponding room is 16.
【Solution 2】
Fifteen is a multiple of 3 and when divided by 7, the remainder is 1. When 7 is divided by 3, the remainder is 1. Therefore, if a number $\overline{ab}$ is printed on the key,

the corresponding room number is the remainder when $a\times7+b\times15$ is divided by $21(=3\times7)$. In the case of two digit number 12, the corresponding room number is $16(1\times7+2\times15=37$，$37=21\times1+16)$.

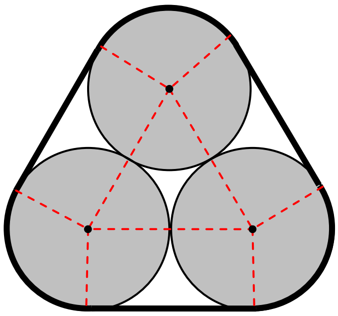
Answer：16

8. Let the positions of the unknown numbers be *A*, *B*, *C*, *D*, *E*, *F*. Since $F+10+3 = F+D+7$，$D = 6$. Also, $+D+B=A+B+7$ implies $A = 3+D = 9$.

A	*B*	7
C	*D*	*E*
F	10	3

Answer：9

9. The cross sectional area can be viewed as three identical rectangles together with three identical circular sectors and one equilateral triangle, as shown in the diagram below.



The length and width of each rectangle is 20 cm and 10 cm respectively. The radius of each sector is10cm and the subtended angle is 120°. The three sectors together form one circle. The length of the side of the equilateral triangle is 20cm. Hence area of the cross section shown in the diagram is $20\times10\times3+10\times10\times\pi+100\sqrt{3}\text{ cm}^2$. And after taking away the areas of the three circles, the total area of the unshaded regions within the rubber band is $600-200\pi+100\sqrt{3}\text{ cm}^2$.

Answer： $600-200\pi+100\sqrt{3}\text{ cm}^2$

10. Suppose *N* is a four-digit number and is a magic number, with the four digits in *N* can be arranged in descending order $a > b > c > d$. From the information given in the question, $\overline{abcd}-\overline{dcba}=N$. From the magnitude of *a*, *b*, *c*, and *d*, the unit digit of *N* is $10+d-a$, tenth digit is $9+c-b$, the hundredth digit is $b-c-1$, and the thousandth digit is $a-d$. From the information given in the question,

$$a+b+c+d=(10+d-a)+(9+c-b)+(b-c-1)+(a-d)=18.$$

As $a-d>b-c-1$, $9+c-b>10+d-a$ and $a-d<a$, we get

$$9+c-b=a \qquad (1)$$

If $10+d-a=b$, then $10+d=a+b$, From (1) we can get $9+c=10+d$, therefore $c=d+1$. Then $a-d=c$, $b-c-1=d$, implies $b=2d+2$, $a=2d+1$, which is $a<b$. This contradicts with $a>b$. Thus $a-d=b$. By $10+d-a>d$, we have $10+d-a=c$, $b-c-1=d$. As a result

$b+c=(a-d)+(10+d-a)=10$, $a+d=8$. Since $a > b > c > d$, we have $a \geq d+3$. Hence the maximum value of d is 2:

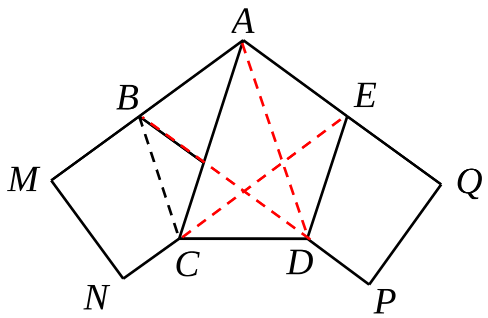
When $d = 1$, we get $a=8-d=7$, $b=a-d=6$, $c=10-b=4$;

When $d = 2$, we get $a=8-d=6$, $b=a-d=4$, $c=10-b=6$, which contradicts with $b > c$.

Therefore, only when $d = 1$, $c = 4$, $b = 6$, $a = 7$ could satisfy the constraints in the question. And the four-digit magic could only be $7641-1467=6174$.

Answer：6174

11. *ABCDE* is a regular pentagon, and *CN*+*DP*=*CD*. The two quadrilaterals *MNCB* and *PQED* could be jointed at the side *MN* and *PQ* by flapping the figure, and this could form a figure which is congruent to the quadrilateral *ACDE*. By analyzing the way in constructing the figure, the regular pentagon *ABCDE* is formed by four congruent quadrilaterals *ABC*, *ACDE*, *ABCD* and *ABDE*. Therefore the area of a rectangle piece of paper is 5 times the area of *ACDE*. Hence the area of *ACDE* is 17.2×2.5÷5=8.6 cm^2.



Answer：8.6 cm^2

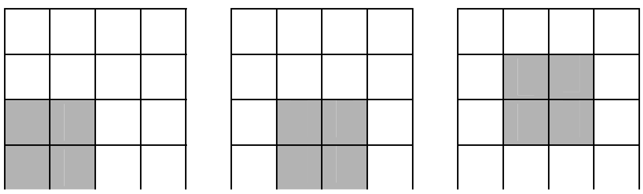
12. There are many variables in the equation, and we can just pay attention to

$$(a_1x+c_1)(a_2x+c_2)=2x^2+7x+6=(x+2)(2x+3).$$

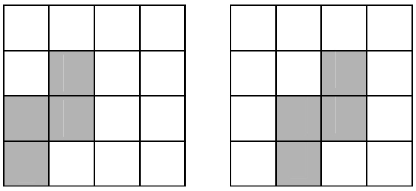
Setting $a_1=1, c_1=2, a_2=2$ and $c_2=3$ (the result may differ by a multiple factor but it will not affect the conclusion). Consider the coefficients of *xy* and *y*, we get $(2x+3)b_1+(x+2)b_2=10x+18$. Solving the quadratic equation, we get $b_1=2, b_2=6$. Hence $M=b_1b_2=12$.

Answer：12

13. According to the symmetric property of square, there are three different settings for block (b) to be placed in a 4x4 square. Block (a) cannot be place in the third setting. If block (a) is put in the two other settings, it will lead to an empty region where no blocks could be placed. Hence, block (a) and block (b) could not be selected simultaneously. Similarly, block (b) and (e) cannot be selected simultaneously.



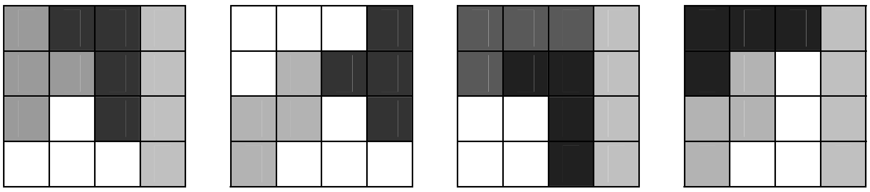
According to the symmetry property of a square, there are two settings for block (e) to be placed in a 4×4 square. Hence, a 4×4 square could not be formed by using block (e) alone. If block (e) is selected, block (d) should also be selected simultaneously.



(1) Only one kind of blocks is used: The first four kinds of blocks can form a 4×4 square (4 pieces each). There are has four choices.

(2) Two kinds of blocks are used together: Block (a) could not be used together with other blocks to form a 4×4 square. Block (e) could only be used together with block (d) to form a 4×4 square. In the remaining three blocks, any two types of blocks could be put together to form a 4×4 square. Hence there are four choices.

(3) Three kinds of blocks are used together: Block (a) and (b) could not be selected at the same time; similarly blocks (b) and (e) could not be selected simultaneously. Hence, there are four choices. They are (a)(c)(d), (a)(d)(e), (b)(c)(d) and (c)(d)(e). The arrangements of blocks are as follows:



(a)(c)(d) (a)(d)(e) (b)(c)(d) (c)(d)(e)

(4) Four kinds of blocks are used together: As block (a) and (b) could not be selected simultaneously, similarly block (b) and (e) could not be selected simultaneously. And block (a) (b) (c) (d) could not form a 4x4 square. Hence, so such arrangement exists.

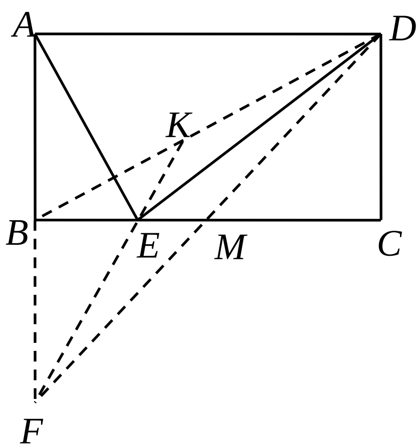
To conclude, there are 12 choices to form a 4×4 square.

Answer：12

14. 【Proof 1】

As shown in the diagram, the M is the mid-point of BC . By symmetry, assume that point E lies on the segment of BM. By extending AB to point F, we have $BF = AB$. **(5 points)**

Obviously, the three points F, B and D are collinear. Connect FE and BD and allow the extension of FE meets BD at the point K. Since $BF=BA$, $BE=BE$, and $\angle FBE = \angle ABE = 90°$, so $\Delta FBE \cong \Delta ABE$, and $FE = AE$. **(5 point)**



If point E is not coincide with points B and M, we get the following through the relations between lengths of the three sides of a triangle：

$$\begin{aligned} AE + DE &= FE + DE \\ &< FE + EK + KD \\ &= FK + KD \\ &< FB + BK + KD \\ &= FB + BD \\ &= AB + BD \end{aligned}$$

If E and M coincide, then $AE + DE = AM + DM = FM + DM = FD$ ；
If E and B coincide, then $AE + DE = AB + BD$.
Hence the largest value of the sum of length of segment AF and DE is $AB + BD$. The required point E is the point B（By symmetry, point C could also be the answer）. **(10 points)**

Answer：The point E required coincided with point B or C.

The Rubric for the marking (points given)

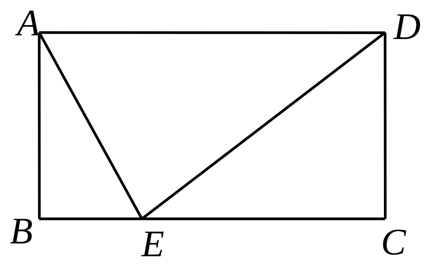
0 point： Conclusion given, but without any explanation.

5 point： Can provide that point A is symmetric of F with reference to the line BC.

10 point： Can provide that point A is symmetric of F with reference to the line BC and prove that $AE = FE$.

10 point： Using the relation of the sides of a triangle and prove that $FB + BD > FE + DE > FD$.

【Proof 2】



The following prove that when point E and B (or point C) are coincide, the sum of the length of the line segment AE and DE is the largest. Let E be any point on the side BC (except the point B or C) . Assume that , $AB=a$, $BC=b$, $BE=x$, $CE=b-x$, from Pythagoras theorem, we have

$AE = \sqrt{a^2 + x^2}$, $DE = \sqrt{a^2 + (b-x)^2}$.

We only need to prove the following:
when $0 < x < b$,

$$\begin{aligned} & \sqrt{a^2 + x^2} + \sqrt{a^2 + (b-x)^2} < a + \sqrt{a^2 + b^2} \\ \Leftrightarrow\ & \sqrt{a^2 + (b-x)^2} - a < \sqrt{a^2 + b^2} - \sqrt{a^2 + x^2} \\ \Leftrightarrow\ & -2a\sqrt{a^2 + (b-x)^2} - 2bx < -2\sqrt{a^2 + b^2}\sqrt{a^2 + x^2} \\ \Leftrightarrow\ & 2\sqrt{a^2 + b^2}\sqrt{a^2 + x^2} - 2bx < 2a\sqrt{a^2 + (b-x)^2} \\ \Leftrightarrow\ & 8b^2x^2 - 8bx\sqrt{a^4 + a^2x^2 + b^2a^2 + b^2x^2} < -8ba^2x \\ \Leftrightarrow\ & 8b^2x^2 + 8ba^2x < 8bx\sqrt{a^4 + a^2x^2 + b^2a^2 + b^2x^2} \end{aligned}$$

$$\Leftrightarrow bx + a^2 < \sqrt{a^4 + a^2x^2 + b^2a^2 + b^2x^2}$$

$$\Leftrightarrow 2ba^2x < a^2x^2 + b^2a^2$$

$$\Leftrightarrow 2bx < x^2 + b^2$$

$$\Leftrightarrow 0 < (x-b)^2$$

The Rubric for the marking (points given)

5 point： Using Pythagoras theorem to provide the length of AE and DE, and provide the first inequality.

5 to 10 point： Provide working process of three steps or more, and can correctly provide calculation and deduction. The 5 to 10 pints are given according to the degree of good working.

15 point： Correct calculation and complete the proof of the first inequality.

15. At most 21 stamps could be cut, as shown in the diagram on the right hand side **(10 points)**.

	X		X		X		X	
		X				X		
	X		X		X		X	
				X				
	X		X		X		X	
		X				X		
	X		X		X		X	

Assume each stamp is a square with length equal to 1. If x stamps are cut, by constraints （1）and（2）, the "perimeter" of remaining stamps is 36+4x. On the other hand, we can take the view that stamps are put back on a frame of stamps (the outer rectangle form by 32 stamps).

Given the "perimeter" of stamps in the frame is (7+9)×4=64, and $7^2 - x = 49 - x$ stamps are needed to put back to the frame. Each time a stamp is put back, the total perimeter is increased by 2. Therefore the perimeter of the remaining stamps is at most $64 + 2(49 - x) = 162 - 2x$.

Thus $36 + 4x \le 162 - 2x$, and $x \le 21$, which implies at most 21 stamps could be cut.

(10 points)

Answer：21

The Rubric for the marking (points given)

10 point： Only the answer and the diagram are given.

10 point： Proof of the conclusion is given.

(5 points for giving the answer 25 by conditions (1) and (2).