Solution Key to Second Round of IMAS 2016/2017
Junior Division

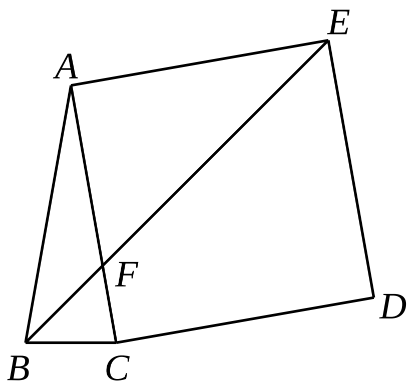
1. Which of the numbers below cannot be expressed as a sum of two prime numbers?
(A) 19 (B) 20 (C) 21 (D) 22 (E) 23

【Suggested Solution】

Note that we can express $19=2+17$, $20=3+17$, $21=2+19$ and $22=3+19$.
Suppose that 23 can be expressed as a sum of two prime numbers, then one must be an odd and the other must be an even. Since the only even prime number is 2, the other number is 21, which is not a prime number, thus, a contradiction. Therefore the answer is (E).

Answer：(E)

2. In $\triangle ABC$, $AB=AC$ and $\angle ACB=80°$. Construct square *ACDE* with the given side *AC*. Lines *BE* and *AC* intersect at point *F*, as shown in the figure. What is the measure of $\angle BFC$?



(A) 55° (B) 60° (C) 65° (D) 70° (E) 75°

【Suggested Solution】

Since $AB=AC$ and $\angle ABC=\angle ACB=80°$, then $\angle BAC=180°-2\times 80°=20°$.
Also, since *ACDE* is a square, then $AE=AC=AB$;

Moreover, $\angle BAE=90°+20°=110°$, therefore $\angle AEB=\dfrac{180°-110°}{2}=35°$.

Thus, $\angle BFC=\angle AFE=180°-90°-35°=55°$.

Answer：(A)

3. Alex and Charles were both sending parcels. The postage rates are as follows: For the first 10 kg and below, the postage price is \$6 per kg; for each successive kilogram after 10 kg, the postage price per kg is slightly lower than that of the first 10kg. It is known that the weight of Alex's parcel is 20% heavier than Charles' parcel, and that the postage prices for Alex and Charles are \$92 and \$80 respectively. How much more is the postage price per kg of the first 10 kg than that of each succeeding kg above 10 kg?
(A) 1.5 (B) 2 (C) 2.5 (D) 3 (E) 3.5

【Suggested Solution 1】

Note that the postage price of a 10 kg parcel is \$60, thus the weight of parcels of Alex and Charles were both more than 10 kg. The postage price for the excess weight above 10 kg paid by Alex was $\$92-\$60=\$32$, while that paid by Charles is $\$80-\$60=\$20$. Suppose the weight of Charles' parcel was x kg, then, the weight of Alex's parcel would be $(1+20\%)x=1.2x$ kg. Then, we have $\dfrac{x-10}{1.2x-10}=\dfrac{20}{32}$, which gives $x=15$. So the postage price for each excess kg above 10 kg is $\$20\div(15-10)=\$20\div5=\$4$, therefore the postage price per kg of the first 10 kg is $\$6-\$4=\$2$ higher than that of each succeeding kg above 10 kg.

【Suggested Solution 2】

Note that the postage price of a 10 kg parcel is \$60, thus the weight of parcels of Alex and Charles were both more than 10 kg. The postage price for the excess weight above 10 kg paid by Alex was $\$92-\$60=\$32$, while that paid by Charles is $\$80-\$60=\$20$. Suppose the cost per kg of the part that is in excess of 10 kg is \$$x$, then Alexs' parcel weights $10+\dfrac{32}{x}$ kg, while Charles' weights $10+\dfrac{20}{x}$ kg. Thus we have $10+\dfrac{32}{x}=1.2\times(10+\dfrac{20}{x})$, which gives $x=4$. The postage price for each excess kg above 10 kg is \$4. Therefore the postal price per kg of the first 10 kg is $\$6-\$4=\$2$ higher than that of each succeeding kg above 10 kg.

Answer：(B)

4. It is given that $A=3x^2+3x$, $B=-x^2+x+5$ and $C=x^2+x-1$. $4A-(B-2(2B-3C)+2A)-2B=?$

(A) $-x^2+x+11$ (B) $-x^2-x+11$ (C) $-x^2+x+1$ (D) $-x^2+x-1$ (E) x^2+x+11

【Suggested Solution】

$4A-(B-2(2B-3C)+2A)-2B=2A+B-6C=B+2(A-3C)=-x^2+x+11$.

Answer：(A)

5. On the bookshelf of Mar, there are Literature, Mathematics, History and Science books. If the number of Mathematics Books is 5 times that of the Literature books, and the number of Science books is 4 times that of the History books, which of the following is not a possible number for the total number of books on the bookshelf?

(A) 21 (B) 23 (C) 26 (D) 29 (E) 30

【Suggested Solution】

Let x be the number of Literature books and y be the number of History books, where x and y are positive integers. From the conditions, we know that the total number of books is $x+5x+y+4y=6x+5y$.

If $x=1$ and $y=3$, then $6x+5y=21$; if $x=3$ and $y=1$, then $6x+5y=23$;
if $x=1$ and $y=4$, then $6x+5y=26$; if $x=4$ and $y=1$, then, $6x+5y=29$.
If $6x+5y=30$, then $6x$ must be a multiple of 5, so x is also a multiple of 5. This will lead to $6x\geq 30$, thus, $5y\leq 0$, a contradiction.

Answer：(E)

6. Fill in the 4×4 box so that the numbers 1, 2, 3, and 4 appear exactly once in each row and column. Referring to the figure below, what is the sum of the values of A and B?

	A	4	
B		1	
1	2	3	4
3	4	2	1

【Suggested Solution】

Notice that the box that is situated below A can't be 1, 2, or 4, thus it must be 3, so $A=1$, while the box situated above box B can't be 1, 3 and 4, thus it must be 2, so we get $B=4$. So $A+B=1+4=5$.

2	1	4	3
4	3	1	2
1	2	3	4
3	4	2	1

Answer： 5

7. The lengths of two sides of a triangle are 6 cm and 13 cm respectively. It is known that the length of the third side is also an integer (in cm). What is the minimum perimeter (in cm) of this triangle?

【Suggested Solution】

According the triangle inequality, the difference of any two sides must be smaller than the third side. We can derive that the smallest possible length of the third side is 8 cm, therefore, the minimum perimeter is $13+6+8=27$ cm.

Answer： 27 cm

8. It is given that Figure 1 shows a circle with diameter of 9 cm. Figure 2 shows an Olympic symbol which consists of five circles, each of diameter 9cm. The distance between two of the tangents to the circles is 4 cm as shown. Find the length of the Olympic symbol (Figure 3)?

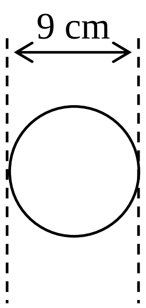


Figure 1

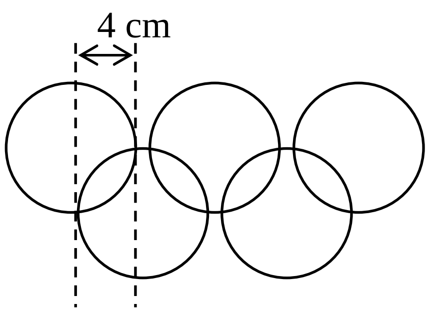


Figure 2

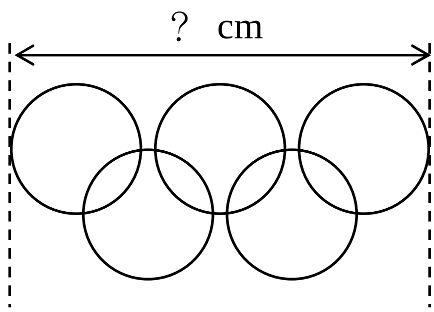


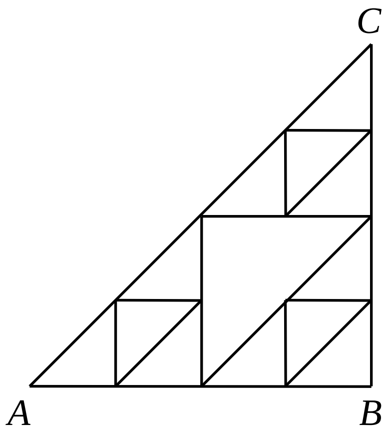
Figure 3

【Suggested Solution】

Observe the 3 circles on top row of the Olympic Symbol. According to Figures 1 and 2, the distance between two adjacent circles is $9-4-4=1$ cm, so the total length of the Olympic Symbol is $3\times 9+2\times 1=29$ cm.

Answer： 29 cm

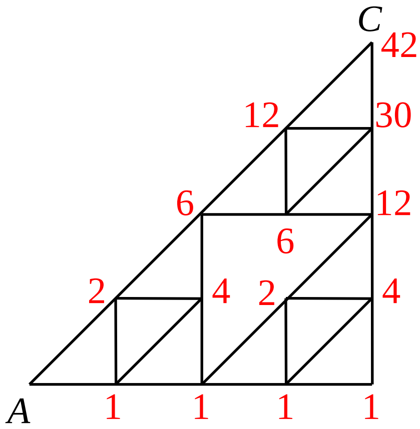
9. The diagram below is composed of many right angled isosceles triangles. Suppose an ant wants to travel from point A to point C, in how many ways can this be done if the ant is only allowed to move up, right or diagonally?



【Suggested Solution】

Refer to the diagram on the right, the total number of ways to go from A to C is 42 .

Answer：42 ways



10. Among the 1000 positive integers from 1 to 1000 inclusive, find the number of positive integers n such that n^3+n^2+n is divisible by 8.

【Suggested Solution】

Note that $n^3+n^2+n=n(n^2+n+1)$. If n is a positive integer, n^2 and n will always have the same number parity, we have n^2+n+1 is always odd. Thus the given expression will only be divisible by 8 if and only if n is divisible by 8. Therefore, the number of possible n is $\frac{1000}{8}=125$.

Answer： 125 numbers

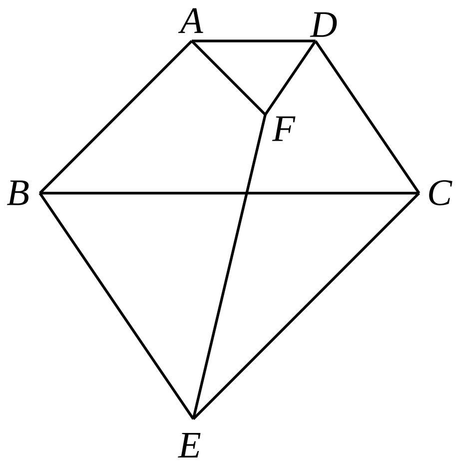
11. Given that $a^2+b^2+c^2=(a+b+c)^2$, where a, b and c are non-zero real numbers. What is the value of $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$?

【Suggested Solution】

Simplifying the expression above, we get $2(ab+bc+ca)=0$, thus, $ab+bc+ca=0$. Since it is known that a, b and c are all non-zero real numbers, then $\frac{ab+bc+ca}{abc}=0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, therefore, $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})-3=-3$.

Answer： -3

12. Refer to the diagram below, in trapezium *ABCD*, *AD*//*BC*. The line passing through *B* and parallel to *CD* intersects the line passing through *C* and parallel to *AB* at point *E*. Point *F* lies inside *ABCD* such that $\angle FAD = \angle ABC$ and $\angle FDA = \angle DCB$. Given that the area of *ABEF* is 20 cm^2 and the area of *DCEF* is 16 cm^2, what is the area of *ABCD*?



【Suggested Solution】

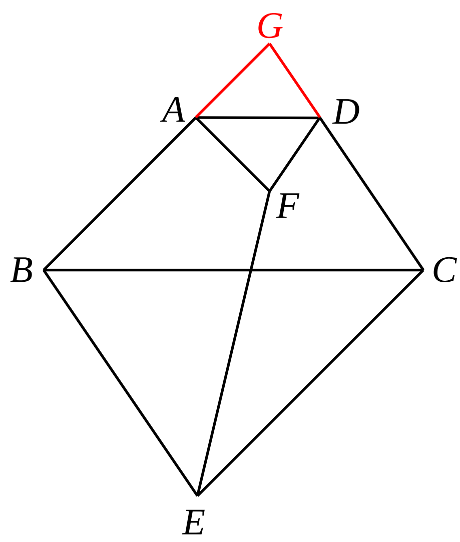
Extend *BA*、*CD* to meet at point *G*, as shown in the right diagram.

Observe that *GBEC* is a parallelogram, therefore $S_{GBEC} = 2S_{GBC}$. Since $\angle FAD = \angle ABC = \angle GAD$, $\angle FDA = \angle DCB = \angle GDA$ and $AD = AD$, we can say that $\triangle FDA \cong \triangle GDA$, therefore $S_{FDA} = S_{GDA}$.

Therefore, $S_{GAFD} = 2S_{GDA}$. So

$$\begin{aligned} S_{ABEF} + S_{DCEF} &= S_{GBEC} - S_{GAFD} \\ &= 2(S_{GBC} - S_{GDA}) \\ &= 2S_{ABCD} \end{aligned}$$

therefore $S_{ABCD} = \dfrac{20+16}{2} = 18$ cm^2.



Answer： 18 cm^2

13. A 4-digit number is said to be ‘good’ if it uses exactly 3 different digits from the set {2, 0, 1, 7} (at most one of the digits used can be repeated). For example, 8712 and 7200 are said to be ‘good’ numbers, while 2017 and 7175 are not. How many ‘good’ numbers are there?

【Suggested Solution 1】

We count the number of ‘good’ numbers in 5 cases.

Case 1: All 4 digits are distinct, and there is no 0 used.

So three of the digits used should be selected from 2, 1 or 7 and one digit should be selected from 3, 4, 5, 6, 8, and 9, therefore, there are 6 ways to choose the four digits. The number of ways to arrange the four digits accordingly is $4\times3\times2\times1 = 24$ ways, therefore, the number of good numbers in this case is $6\times24 = 144$.

Case 2: All 4 digits are distinct and 0 is used.

Since 0 is used, two of the digits used should be selected from 2, 1 or 7, and one digit to be selected from 3, 4, 5, 6, 8, and 9, so there are $3\times6=18$ ways to choose the four digits. The number of ways to arrange the four digits is $3\times3\times2\times1=18$ ways, therefore, the total number of good numbers in this case is $18\times18=324$.

Case 3: Digits with repetition and there is no 0 used.

Then all digits must be 2, 1 or 7. There are 3 ways to select a repeating number from 2, 1 or 7. The number of ways to arrange the four digits is $\dfrac{4\times3\times2\times1}{2}=12$ ways.

Therefore, the total number of good numbers in this case is $3\times12=36$.

Case 4: Digits with repetition and 0 is used once.

Select two digits from 2, 1 or 7, one will be used twice and the other is used once, so there are $3\times2=6$ ways in doing so. The number of ways in arranging the four digits (since 0 cannot be the thousands digit) is $\dfrac{3\times3\times2\times1}{2}=9$. Therefore, the total number of good numbers in this case is $6\times9=54$.

Case 5: Digits with repetition and 0 is used twice.

There are 3 ways to select two digits from 2, 1 and 7. The number of ways in arranging the four digits is $\dfrac{2\times3\times2\times1}{2}=6$. Therefore, the number of good numbers in this case is $3\times6=18$.

Therefore the total number of good numbers is $144+324+36+54+18=576$.

【Suggested Solution 2】

We first find all the possible number of ways of forming the 4-digits numbers that satisfies that condition of the problem , and then subtracting the cases where 0 is in the thousands place.

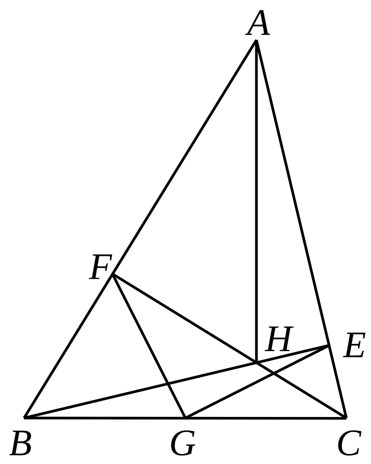
If all four digits are different, we select three digits from the set 2, 0, 1, 7 and select one digit from 3, 4, 5, 6, 8 and 9, therefore, there are a total of $4\times6\times4\times3\times2\times1=576$ ways in doing such. Now, we subtract the cases where 0 is found to be in the thousands place: if this is the case, then 2 digits should be selected from 2, 1 or 7, while the 3rd digit should be from 3, 4, 5, 6, 8 and 9, therefore there is a total of $3\times6\times3\times2\times1=108$ numbers when the thousands digit is 0.

Next, we consider the case when there is repeated use of digits. Select three digits from 2, 0, 1 or 7 , one will be used twice and the other two are used once. You have 4 ways to select the repeated digit and 3 ways to select the other two digits that will appear once. There are $\dfrac{4\times3\times2\times1}{2}=12$ ways in selecting such. So in this case, there are $12\times12=144$ different ways. Then we again remove the instance wherein 0 will be in the thousands place, which is one-fourth of the cases, therefore, $\dfrac{144}{4}=36$ ways.

Therefore the total number is $576-108+144-36=576$ ways.

Answer： 576 numbers

14. In $\triangle ABC$, point G is the midpoint of segment BC, $BE \perp AC$, $CF \perp AB$ and lines BE and CF intersects at point H. If $\angle EGF = 90°$, prove that $AH = BC$.



【Suggested Solution】

Since $BE \perp AC$, $CF \perp AB$ and G is the midpoint of BC, therefore, $GB = GC = GE = GF$. (5 points)

Because of this, $\angle FCG = \dfrac{180° - \angle FGC}{2}$,

$\angle ECG = \dfrac{180° - \angle EGC}{2}$, therefore

$$\angle ECF = \frac{180° - \angle EGC}{2} - \frac{180° - \angle FGC}{2} = \frac{\angle FGC - \angle EGC}{2} = \frac{\angle EGF}{2} = 45°$$ (5 points)

Since $CF \perp AB$, therefor AFC is an isosceles right triangle, therefore $AF = CF$ (5 points)

Since point H is the orthocenter of $\triangle ABC$, we can see that $AH \perp BC$, $\angle FAH = 90° - \angle ABC = \angle FCB$ and since $\angle AFH = 90° = \angle CFB$, $AF = CF$, we know that $\triangle AFH \cong \triangle CFB$. Thus, $AH = BC$ (5 points)

15. It is known that the equation $x^2 + (x+k)^2 = y^2$ has positive integers solutions (x, y), where x and y are relatively prime. If k is a positive integer greater than 1, what is the minimum value of k?

【Suggested Solution】

When $k = 7$, the expression has a solution (5, 13). Below proves that $k = 7$ is the minimum value. (5 points)

First note that x and k must both be relatively prime, otherwise if they will have a common factor p, then $p \mid y^2$, therefore $p \mid y$, then x and y will not be relatively prime, then it will be a contradiction.

Suppose k is an even number, then x must be odd, so x^2 and $(x+k)^2$ will have a remainder of 1 when divided by 4, while y^2 will have a remainder of 2 when divided by 4, which is impossible. (5 points)

Suppose $3 \mid k$, then x is not a multiple of 3, so x^2 and $(x+k)^2$ will have a remainder of 1 when divided by 3, while y^2 will have a remainder of 2 when divided by 3, which is impossible. (5 points)

Suppose $5 \mid k$, then x is not a multiple of 5, so x^2 and $(x+k)^2$ will have the same remainder of 1 or 4 when divided by 5, while y^2 will have a remainder of 2 or 3 when divided by 5, which is impossible.

In summary, k cannot be multiples of 2, 3 or 5, since $k > 1$, the smallest possible value of $k = 7$. (5 points)

【Note】Gave a set of solutions and declared that $k = 7$ is the minimum will merit 5 points, Stated that k cannot be equal to 2, 3 and 5 will also merit 5 points.