Solution Key to Second Round of IMAS 2017/2018
Junior Division

1. Among all the expressions listed below, how many are negative numbers?

$$(1000-1)^1,\ (1000-2)^2,\ \ldots,\ (1000-n)^n,\ \ldots,\ (1000-2018)^{2018}.$$

（A）509 （B）510 （C）1009 （D）1018 （E）1019

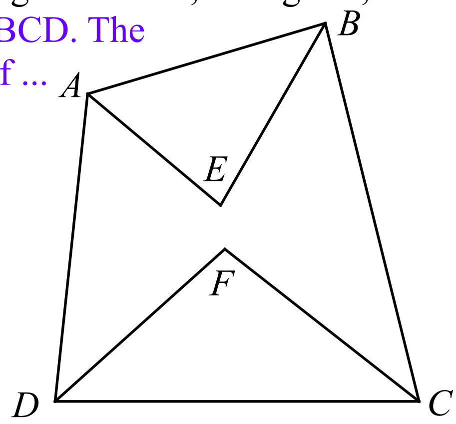
【Solution】

Since even powers of a real number are always non-negative, odd powers of a negative number are always negative and any powers of a positive number are always positive, $(1000-n)^n$ is negative for $1000-n$ being negative and n being odd, thus, n should be odd and over 1000. There are $\frac{2018-1000}{2}=509$ negative numbers.

Answer：(A)

2. In convex quadrilateral *ABCD*, bisectors of $\angle DAB$ and $\angle ABC$ intersect at *E*, bisectors of $\angle BCD$ and $\angle CDA$ intersect at *F*, as shown in the figure below. If $\angle AEB=80°$, what is the angle measure, in degrees, of $\angle DFC$?

Given a convex quadrilateral ABCD. The bisectors of ….., the bisectors of ...



（A）80 （B）90 （C）100

（D）110 （E）Undetermined.

【Solution 1】

Since $\angle AEB=180°-(\frac{\angle DAB+\angle ABC}{2})$ and $\angle DFC=180°-(\frac{\angle CDA+\angle BCD}{2})$, we have $\angle AEB+\angle DFC=360°-(\frac{\angle DAB+\angle ABC+\angle CDA+\angle BCD}{2})$.

And the sum of the inner angles of a quadrilateral is $360°$,
so $\angle AEB+\angle DFC=360°-180°=180°$ and hence $\angle DFC=180°-80°=100°$.

【Solution 2】

$\angle AEB=180°-(\frac{\angle DAB+\angle ABC}{2})$, $\angle DFC=180°-(\frac{\angle CDA+\angle BCD}{2})$,

thus $\angle DAB+\angle ABC=2\times(180°-80°)=200°$. Since the sum of the inner angles of a quadrilateral is $360°$, one has $\angle CDA+\angle BCD=360°-200°=160°$, then

$\angle DFC = 180° - \frac{160°}{2} = 100°$.

Answer: (C)

3. Two numbers m and n, which may be equal, are taken from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Which number below is not a possible value of $10(m+n)-mn$?

(A) 19 (B) 55 (C) 72 (D) 79 (E) 83

【Solution】

Since $10(m+n)-mn = 100-(10-m)(10-n)$, $9 \ge 10-m \ge 1$ and $9 \ge 10-n \ge 1$, $10(m+n)-mn$ is less than 100, and its difference with 100 is $(10-m)(10-n)$, which are the product of two digits. Observe that $100-19=81=9\times 9$, $100-55=45=9\times 5$, $100-72=28=4\times 7$, $100-79=21=3\times 7$, but $100-83=17$ can not express as a product of two digits.

Answer: (E)

4. If a and b are real numbers, which of the following expressions below must be non-negative?

(A) a^2+b^2+a+b (B) $a^{2018}+b^{2017}$ (C) $a^4b^4+a^2b^2-1$ (D) $a^3b^3-2a^2b^2+ab$ (E) $a^2b^2+2ab+1$

【Solution】

When $a=0$, $b=-\frac{1}{2}$, the value of (A) is $-\frac{1}{4}$; when $a=0$, $b=-1$, the value of (B) is -1; when $a=0$, $b=0$, the value of (C) is -1; when $a=1$, $b=-1$, the value of (D) is -4; but $a^2b^2+2ab+1=(ab+1)^2 \ge 0$.

Answer: (E)

5. The product of the sum and arithmetic mean of n integers is 2018. Which of the following statements below is true?

(A) Minimum of n is 1 (B) Minimum of n is 2 (C) Minimum of n is 1009 (D) Minimum of n is 2018 (E) No such n exists.

【Solution】

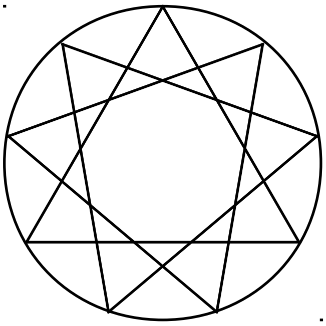
Denote by S as the sum of the n integers. Then $S\times\frac{S}{n}=2018$, so $S^2=2018n$. Since $2018=2\times 1009$ is a product of distinct primes, 2018 divides S and hence 2018^2 divides S^2. Thus 2018 divides n, i.e. $n \ge 2018$. On the other hand, take $S=n=2018$, $S^2=2018\times 2018=2018n$ is a solution to the problem.

Be careful, not "divides" - but "is a divisor of"

Answer: (D)

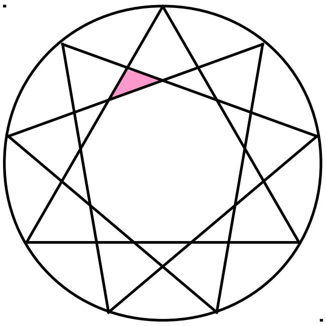
6. Rotate an equilateral triangle inscribed in a circle 40 degrees clockwise and counter-clockwise, as shown in the figure below. How many triangles are there in

the figure?

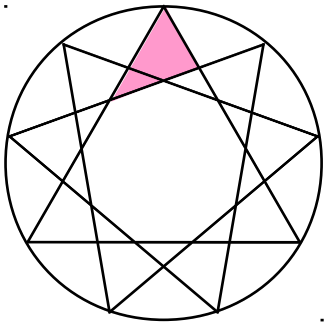


【Solution】

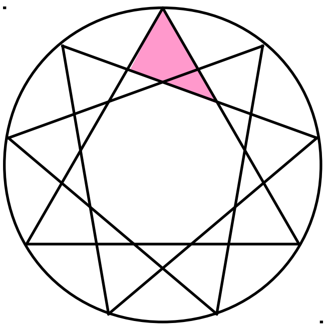
(i) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



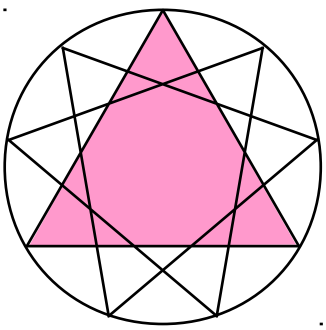
(ii) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



(iii) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



(iv) There are 3 triangles of same size but in different positions as the shaded triangle in the figure below.



Totally there are $9+9+9+3=30$ triangles.

Answer: 30.

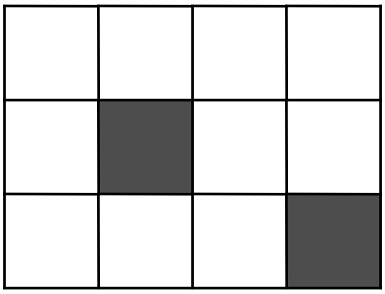
7. Consider a four-digit number $\overline{abcd}$ where a and d are both non-zero. If the last two digits in the sum of $\overline{abcd}$ and $\overline{dcba}$ are 58, what is the maximum possible value of $\overline{abcd}$?

【Solution】

If a and b are not both 9, then $\overline{abcd} \le 9899$. If $a=b=9$, then $a+d=18$ and $b+c+1=15$. Thus, $d=9$ and $c=5$, and the maximum value of $\overline{abcd}$ is 9959.

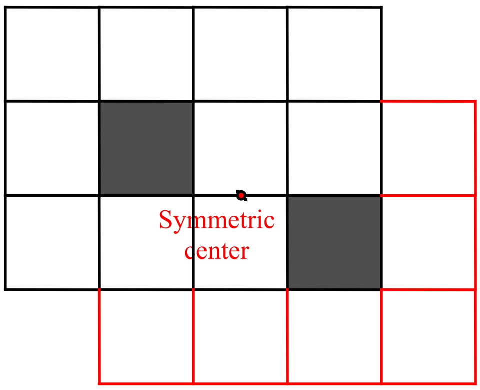
Answer: 9959

8. A rectangle is divided into 12 unit squares such that 10 are white and 2 are black, as shown in the figure below. To form a centrally symmetric picture by adding some white squares but no black squares, what is the least number of white squares needed?



【Solution】

Since no more black unit squares are added, the symmetric center of the whole picture is the symmetric center of the two black unit squares. 4 white unit squares are already symmetric to one another with respect to this center. 6 more white unit squares are needed to be symmetric to those 6 alone white unit squares, as in the figure to the right.



Answer: 6

9. A three-digit number is said to be "lucky" if it is divisible by 6 and by swapping its last two digits will give a number divisible by 6. How many "lucky" numbers are there?

【Solution】

Being divisible by 6 is equivalent to being divisible by both 2 and 3. Thus, the last two digits of a lucky number is even. A number is divisible by 3 if and only if the sum of all digits is divisible by 3. The remainder divided by 3 of the first number of the lucky number is determined by the last two digits. For any remainder, there are exactly 3 non-zero digits. Thus, the number of all lucky numbers is $5\times5\times3=75$.

Answer: 75

10. Find the value of x such that both x and $\sqrt{2017-99\sqrt{x}}$ are integers.

【Solution】

It is obvious that $\sqrt{x}$ should be an integer and $2017-99\sqrt{x}$ is a perfect square. Since $2017-99\sqrt{x}\geq0$, $\sqrt{x}\leq\frac{2017}{99}<21$. The possible values of $\sqrt{x}$ are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. Similarly, $2017-99\sqrt{x}$ are respectively 2017, 1918, 1819, 1720, 1621, 1522, 1423, 1324, 1225, 1126, 1027, 928, 829, 730, 631, 532, 433, 334, 235, 136, 37. Only for $\sqrt{x}=8$, we get a perfect square $2017-99\sqrt{x}=1225$. Thus $x=8^2=64$.

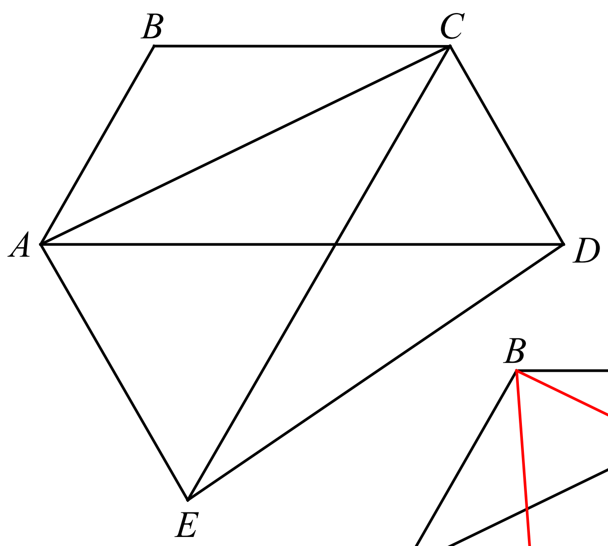
Note that by consider the modulo of 4 → only need to consider the sqrt of x is 4k or 4k+1.

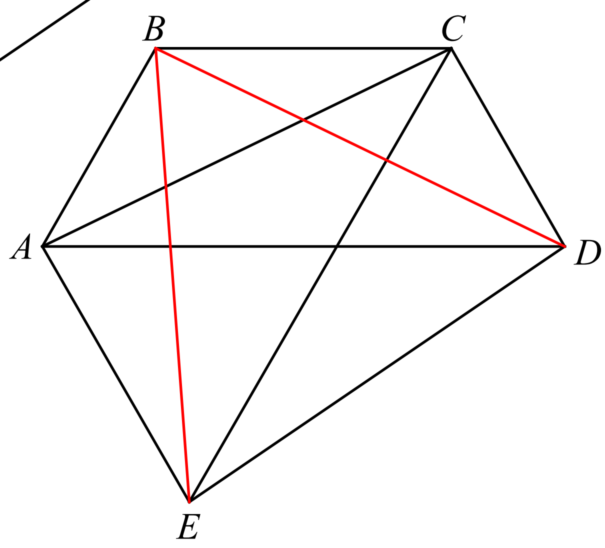
Consider the modulo of 5 → only need to consider the sqrt of x is 5h+2 or 5h+3 or 5h+4.

So maybe we only need to check 4,5,8,9,12,13,17,20

Answer: 64

11. In the figure below, quadrilaterals $ABCD$ and $ABCE$ are both isosceles trapezoids, where $AB//CE$ and $BC//AD$. If $AC=DE$, what is the measure, in degrees, of $\angle ABC$?





【Solution】

Connect BE and BD. Since the diagonals of an isosceles trapezoid are equal, one knows that $BE = AC = BD$. By $AC = DE$, one knows that BDE is an equilateral triangle and $\angle EBD = 60°$. Now

$$\begin{aligned}\angle EBD &= \angle ABC - \angle ABE - \angle DBC \\ &= \angle ABC - \angle BAC - \angle BCA \\ &= 2\angle ABC - 180° = 60°\end{aligned}$$

One gets $\angle ABC = 120°$.

Answer: 120°

12. Place $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, …, $\sqrt{100}$ into several groups such that the sum of each group is not more than 10. Find the least number of groups needed to attain this kind of an arrangement?

【Solution】

Since $\sqrt{25}+\sqrt{26} > 5+5 = 10$， $\sqrt{25}$, $\sqrt{26}$, $\sqrt{27}$, …, $\sqrt{100}$ are arranged into different groups. So there are at least 76 groups. One the other hand, arrange $\sqrt{25-n}$ and $\sqrt{25+n}$ into one group for $n = 1$, 2, …, 24 and each remaining numbers into one group. Since $(\sqrt{25+n}+\sqrt{25-n})^2 = 50 + 2\sqrt{25^2-n^2} < 100$, this gives a minimum number of 76 groups that is required to attain such arrangement.

Answer: 76

13. There is a sequence of five positive integers. Each number right after the first term is at least twice the number before it. If the sum of the five numbers is 2018, what is the least possible value of the last number?

Each term after the first term is at least twice the previous term.

【Solution】

Let x be the last number. Then the first four numbers are at most $\frac{x}{16}$, $\frac{x}{8}$, $\frac{x}{4}$ and $\frac{x}{2}$.

Thus $\frac{x}{16}+\frac{x}{8}+\frac{x}{4}+\frac{x}{2}+x \geq 2018$, i.e. $x \geq \frac{2018\times 16}{31} = 1041\frac{17}{31}$, so x is at least 1042.

Taking the five numbers as 65, 130, 260, 521 and 1042 satisfies the requirement of the problem.

Answer: 1042

14. Let a, b, c and d be four positive integers such that $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$ are simplified fractions and $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}$ is an integer. Prove that $d \geq a-1$.

【Solution】

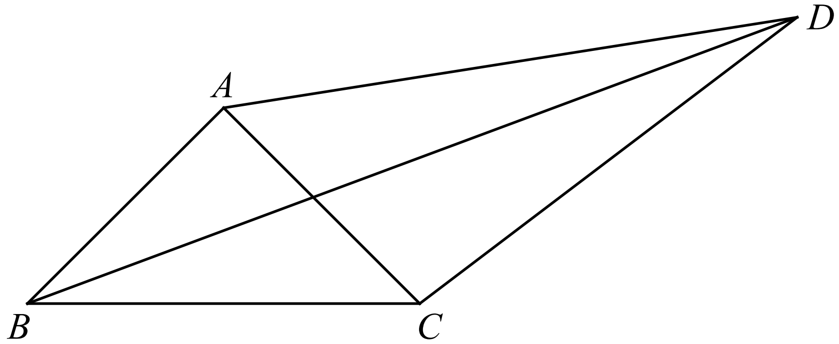
From the condition of the problem, we know that b is coprime to a and c and c is coprime to b and d. (5 points)

Since $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}$ is an integer, $ac(\frac{b}{a}+\frac{c}{b}+\frac{d}{c})=bc+ad+\frac{ac^2}{b}$ is also an integer. So $\frac{ac^2}{b}$ is an integer. Since b is coprime to a and c, $b=1$. (5 points)

$\frac{1}{a}+\frac{d}{c}$ is then an integer, since both are reduced fractions, $a=c$. (5 points)

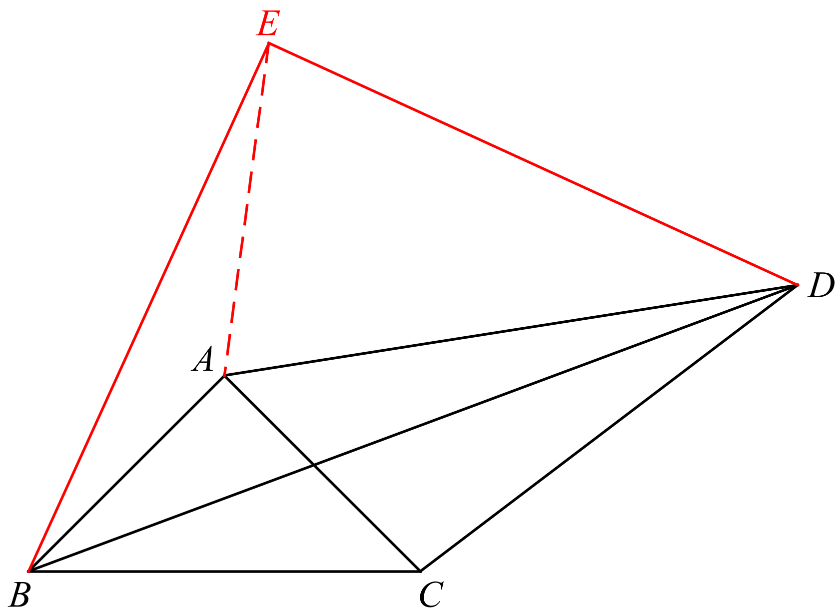
$\frac{d+1}{a}$ is then an integer, $d+1$ is a multiple of a, $d+1\geq a$ and $d\geq a-1$. (5 points)

15. In the figure below, ABC is a right isosceles triangle where $AB=AC$. Let D be an exterior point such that $BD=\sqrt{2}AD$. Prove that $\angle ADC+\angle BDC=45°$.



【Solution】

Draw the right isosceles triangle EBD so that $\angle BED=90°$ and E, C are on the opposite sides of BD, as shown in the figure below.



$\frac{BD}{BC} = \frac{\sqrt{2}BE}{\sqrt{2}BA} = \frac{BE}{BA}$ and $\angle EBA = 45^\circ - \angle ABD = \angle DBC$ implies that $\triangle EBA \sim \quad DBC$. (10 points)

So, one gets $\angle BDC = \angle BEA$. Thus $DE = \frac{BD}{\sqrt{2}} = DA$ implies $\angle DEA = \angle DAE$ and

$$\angle EDA = 180^\circ - 2\angle DEA = 2(90^\circ - \angle DEA) = 2\angle BEA = 2\angle BDC.$$ (5 points)

Then $\angle ADC + \angle BDC = \angle ADB + 2\angle BDC = \angle ADB + \angle EDA = 45^\circ$. (5 points)