Solution to
Fifth International Mathematics Assessment for Schools
Round 1 of Junior Division

1. What is the value of $\sqrt{(-20)^2}+16^2-15^2$?

   (A) -19　(B) 11　(C) 21　(D) 51　(E) 61

【Solution】

$\sqrt{(-20)^2}+16^2-15^2=20+256-225=51$.

Answer：(D)

2. The table below summarizes the results of a test in a certain class. What is the total score of this class?

Summary of the results of a test			
No. of students	The highest score	The lowest score	The average score
42	100	16	84.5

   (A) 672　(B) 3528　(C) 3549　(D) 4200　(E) 4872

【Solution】

The total score of this class is $84.5\times 42=3549$.

Answer：(C)

3. A three-digit number is not divisible by 24. When divided by 24, the quotient is a and the remainder is b. What is the minimum value of $a+b$?

   (A) 5　(B) 6　(C) 7　(D) 8　(E) 9

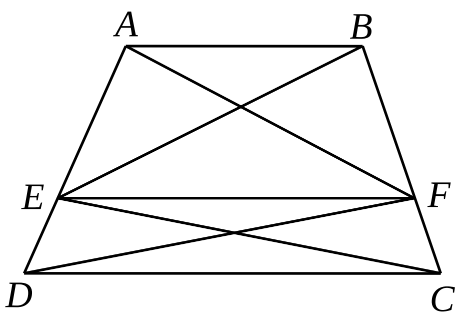
【Solution】

Observe that $100=4\times 24+4$. When the dividend increases, the quotient $a\geq 4$. If $a=4$, then the remainder $b\geq 4$ and hence $a+b\geq 8$. If $a\geq 5$, then $b\geq 1$ since the three-digit is not a multiple of 24. Thus $a+b\geq 6$. When the three-digit number is 121, we have $a=5$, $b=1$ and $a+b=6$.

Answer：(B)

4. In the trapezium $ABCD$, AB is parallel to CD. E and F are points on AD and BC respectively such that EF is also parallel to AB. The area, in cm^2, of triangles BAF, CDF and BCE are 8, 7 and 18 respectively. What is the area, in cm^2, of $ABCD$?

   (A) 30　(B) 32　(C) 33
   (D) 35　(E) 36



【Solution】

The area of triangle BAE is equal to the area of triangle BAF since $EF//AB$. And the area of triangle CDE is equal to the area of triangle CDF since $EF//CD$. So the area of trapezium $ABCD$ is equal to the sum of the areas of triangle BAE, CDE and BCE, which is $8+7+18=33\text{cm}^2$.

Answer：(C)

5. What is the value of the negative number x which satisfies $|x-3|=|3x|+1$?

(A) -2 (B) -1 (C) $-\frac{2}{3}$ (D) $-\frac{1}{2}$ (E) $-\frac{1}{4}$

【Solution】

Since x is a negative number, $x-3<0$ and $3x<0$. Thus the original equation can be simplified to $3-x=-3x+1$. So $x=-1$.

Answer：(B)

6. The radius of each wheel of Rick's bicycle is 25 cm. He rides to school at a constant speed and arrives after 10 minutes. During this time, each wheel makes 160 revolutions. Of the following five distances, which is closest to that between Rick's home and school?

(A) 1 km (B) 1.5 km (C) 1.8 km (D) 2 km (E) 2.5 km

【Solution】

In order to estimate the distance between Rick's home and school, take $\pi=3.14$. When the wheel of Rick's bicycle is turned one round, Rick's bicycle goes ahead about $2\times3.14\times25=157$ cm. So the distance between Rick's home and school is $157\times160\times10=251200\text{cm}=2512\text{m}$, which is about 2.5 km.

Answer：(E)

7. How many 2-digit numbers are there such that at least one digit is divisible by 3?

(A) 48 (B) 54 (C) 60 (D) 66 (E) 80

【Solution 1】

If the tens-digit is one of 3, 6 and 9, then such a two-digit number satisfies the condition. There are 30 such two-digit numbers.

If the tens-digit is one of 1, 2, 4, 5, 7 and 8, then the units-digit must be one of 0, 3, 6 and 9. There are $6\times4=24$ such two-digit numbers.

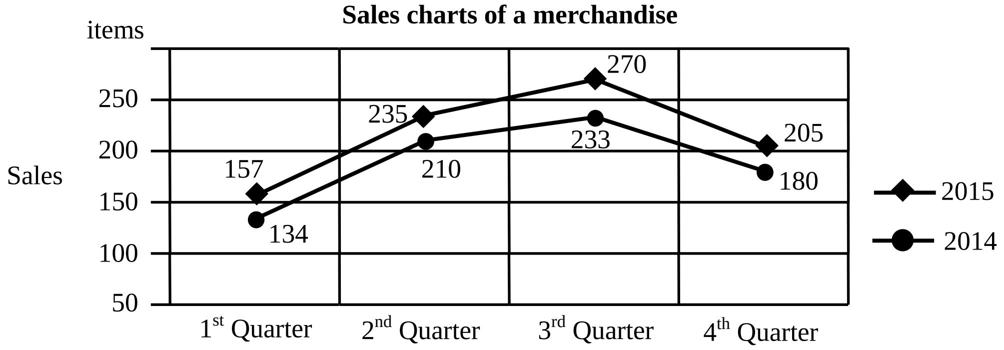
So there are totally $30+24=54$ two-digit numbers satisfying the condition.

【Solution 2】

Observe that the two digits of the two-digit numbers which don’t satisfy the condition are both one of 1, 2, 4, 5, 7 and 8. So there are $6\times6=36$ such two-digit numbers. Since there are totally 90 two-digit numbers, there are totally $90-36=54$ two-digit numbers satisfying the condition.

Answer：(B)

8. The chart below shows the sale figures of a certain merchandise in 2014 and 2015 by the season. How many more items were sold in 2015 than in 2014?



(A) 23 (B) 48 (C) 85 (D) 90 (E) 110

【Solution 1】

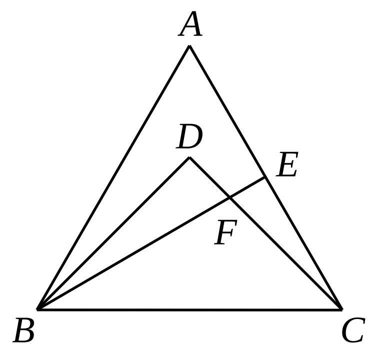
$157+235+270+205=867$ items were sold in 2014 and $134+210+233+180=757$ in 2015. So $867-757=110$ more items were sold in 2015 than in 2014.

【Solution 2】

$157-134=23$ more items were sold in the first quarter of 2015 than in the first quarter of 2014, $235-210=25$ more items were sold in the second quarter of 2015 than in the second quarter of 2014, $270-233=37$ more items were sold in the third quarter of 2015 than in the third quarter of 2014 and $205-180=25$ more items were sold in the fourth quarter of 2015 than in the fourth quarter of 2014. So $23+25+37+25=110$ more items were sold in 2015 than in 2014.

Answer：(E)

9. *ABC* is an equilateral triangle. *D* is a point inside such that *BCD* is a right isosceles triangle. The altitude *BE* of *ABC* intersects *CD* at *F*. What is the measure, in degrees, of $\angle CFE$?



(A) 75°　(B) 70°　(C) 65°　(D) 60°　(E) 55°

【Solution】

$\angle BCD=\dfrac{180°-90°}{2}=45°$ since $BD=CD$ and $BD\perp CD$. Because triangle *ABC* is an equilateral triangle, $\angle BCA=60°$. Thus $\angle DCA=60°-45°=15°$. Now, since $BE\perp AC$, $\angle CFE=90°-15°=75°$.

Answer：(A)

10. In how many ways can 36 be expressed as the sum of two prime numbers, the first larger than the second?

(A) 1　(B) 2　(C) 3　(D) 4　(E) 5

【Solution】

Observe that the larger prime number should be between 18 and 35. So the larger prime number is 19, 23, 29 or 31 and the other prime number is 17, 13, 7 or 5, respectively. There are totally 4 ways.

Answer：(D)

11. Every student in a class is either in the mathematics club or the language club, and one third of them are in both. If there are 22 students in the language club, 4 less than the number of students in the mathematics club, how many students are there in this class?

(A) 12　(B) 18　(C) 24　(D) 30　(E) 36

【Solution 1】

From the conditions, there are $22+4=26$ students in the mathematics club. Since one third of them are in both clubs, the sum of the numbers of students in the mathematics club and in the language club is equal to four third of the number of students in this class. So there are $(22+26)\div\dfrac{4}{3}=36$ students in this class.

【Solution 2】

From the conditions, there are $22+4=26$ students in the mathematics club. Suppose there are x students in the mathematics club and the language club, then there are $3x$ students in this class. So $22+26-x=3x$. Thus $x=12$ and hence there are $12\times 3=36$ students in this class.

Answer：(E)

12. The average of a group of numbers is 5. A second group contains twice as many numbers and its average is 11. What is the average when the two groups are combined?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

【Solution 1】

Since the number of the second group is twice the number of the first group, we can assume there are 2 numbers in the second group and 1 number in the first group. Thus the average when the two groups are combined is $\dfrac{5+11\times 2}{1+2}=9$.

【Solution 2】

Suppose there are k numbers in the first group and $2k$ numbers in the second group. Then the sum of the first group is $5k$ and the sum of the second group is $11\times 2k=22k$. Thus the average when the two groups are combined is $\dfrac{5k+22k}{k+2k}=9$.

Answer：(D)

13. What is the value of x^y if $\sqrt{x-1}+\sqrt{1-x}+y=2016$?

(A) 2015 (B) 2016 (C) $\dfrac{1}{2016}$ (D) 1 (E) 0

【Solution】

Observe that $x-1$ and $1-x$ are both non-negative numbers, otherwise $\sqrt{x-1}$ or $\sqrt{1-x}$ are meaningless. Since $x-1$ is the additive inverse of $1-x$, $x-1=1-x=0$. So $x=1$ and hence $y=2016$. Thus $x^y=1^{2016}=1$.

Answer：(D)

14. Each of A and B goes to the gymnasium 3 or 4 times a week. After n weeks, A has been there 57 times while B has been there only 47 times. What is the value of n?

(A) 15 (B) 16 (C) 17 (D) 18 (E) 19

【Solution】

From the conditions, we can conclude $3n\le 47$ and $4n\ge 57$, so $14\frac{1}{4}\le n\le 15\frac{2}{3}$.

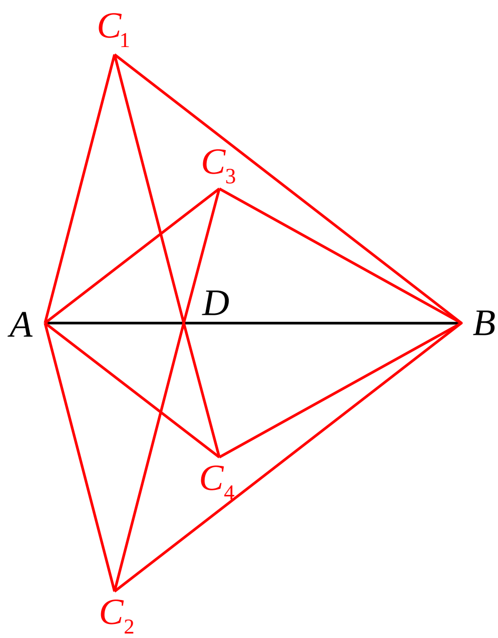
Since n is a positive integer, $n=15$.

Answer：(A)

15. D is a point on AB such that $AD=1$ and $BD=2$. How many points C are there in the plane such that both ACD and BCD are isosceles triangles?

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

【Solution】



Observe that $\angle ADC$ and $\angle BDC$ are supplementary angles, so one of them is not an acute angle and it must be a vertex angle of an isosceles triangle.

If $\angle BDC$ is a vertex angle of an isosceles triangle, then $DC = 2$. Since the sum of the lengths of any two sides of a triangle must be greater than the length of the third side and the difference of the lengths of any two sides of a triangle must be less than the length of the third side, $1 < AC < 3$ and hence $AC = DC = 2$. There are two such points, C_1 and C_2 as shown in the figure.

If $\angle ADC$ is a vertex angle of an isosceles triangle, then $DC = 1$. Since the sum of the lengths of any two sides of a triangle must be greater than the length of the third side and the difference of the lengths of any two sides of a triangle must be less than the length of the third side, $1 < BC < 3$ and hence $BC = DB = 2$. There are two such points, C_3 and C_4 as shown in the figure.

So there are 4 points satisfying the conditions.

Answer：(B)

16. From a 5×5 square piece of paper, two 2×4 rectangles are cut off along the grid lines. In how many different ways can this be done?

(A) 6 (B) 9 (C) 12 (D) 18 (E) 24

【Solution】

Observe that both of the two rectangles should be 2×4 or 4×2. Consider both of them are 2×4 first. If one of them lies on the first and second rows, then the other one should lie on third and fourth rows, or fourth and fifth rows. If one of them lies on the second and third rows, then the other one should lie on fourth and fifth rows. So there are 3 different situations. In each situation, there are 2 locations for each rectangle. So there are $3\times2\times2=12$ different ways. By symmetry, we know there are also 12 different ways if both of the two rectangles are 4×2. Thus there are totally $12+12=24$ different ways.

Answer：(E)

17. The number a is 5 more than its reciprocal. What is the value of $(a^2-1)^2-125a$?

(A) 5 (B) 25 (C) 125 (D) $\dfrac{1+\sqrt{21}}{2}$ (E) $5\sqrt{21}$

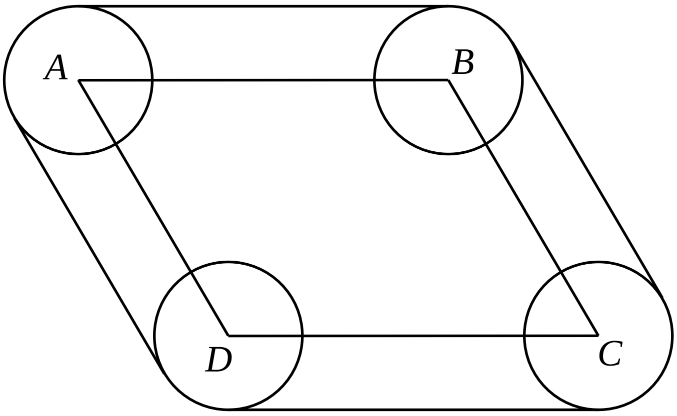
【Solution】

Observe that $a-\dfrac{1}{a}=5$. Since a is not 0, $a^2-1=5a$, i.e., $a^2-5a=1$. So

$$(a^2-1)^2-125a=(5a)^2-125a=25(a^2-5a)=25.$$

Answer：(B)

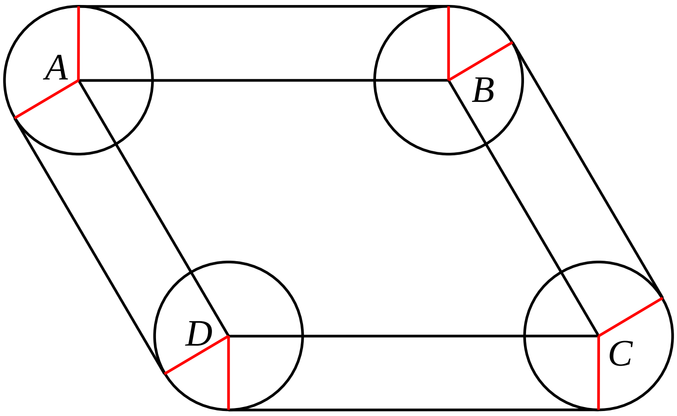
18. With each vertex of a parallelogram $ABCD$ as centre, a circle is drawn. Exterior common tangents are then drawn, as shown in the diagram below. If the perimeter of $ABCD$ is 36 cm and the radius of each circle is 2 cm, what is the maximum area, in cm^2, of the figure enclosed by the circular arcs and tangents?



(A) $117+4\pi$ (B) $144+4\pi$ (C) $153+4\pi$ (D) $144+12\pi$ (E) $153+12\pi$

【Solution】

For each circle, draw the segments from the center to tangent point, as shown in the figure. Thus the figure enclosed by the circular arcs and tangents is divided into the original parallelogram, four rectangles and four sectors. Since the center of a circle is the vertex of a right angle of each of two rectangles, the vertex of an interior angle of the original parallelogram and the center of a sector, the central angle of the sector and the interior angle of the original parallelogram in the same circle are supplementary angles. Thus we can conclude that the sum of the areas of four sectors is equal to the area of a circle, which is $4\pi\,cm^2$ because the sum of the angles of the interior angles of a parallelogram is $360°$. Observe that the sum of the areas of four rectangles is equal to the radius of a circle multiplied by the perimeter of the parallelogram, which is $2\times 36=72\,cm^2$. When we fixed the perimeter of the parallelogram, the maximum area of the parallelogram will occur as the parallelogram is exactly a square and the maximum value is $\left(\frac{36}{4}\right)^2=81\,cm^2$. Thus the maximum area of the figure enclosed by the circular arcs and tangents is $4\pi+72+81=153+4\pi\,cm^2$.



Answer：(C)

19. What is the smallest positive integer with 12 positive divisors such that it is relatively prime to (2016^3-2016)?

(A) 7007 (B) 9163 (C) 26741 (D) 39083 (E) 52877

【Solution】

$2016^3-2016=2016\times(2016^2-1)=2015\times2016\times2017=2^5\times3^2\times5\times7\times13\times31\times2017$, so the prime factors of the positive integer which satisfy the conditions should be 11, 17, 19, ⋯, 29, 37, ⋯, 2011, 2027, ⋯. Since the positive integer has 12 positive divisors, it is of the form p^{11}, p^5q, p^3q^2 or p^2qr, where *p*, *q* and *r* are different prime numbers. It is obviously that the first three forms are all greater than 10^5 and the smallest value of the last one is $11^2\times17\times19=39083<10^5$.

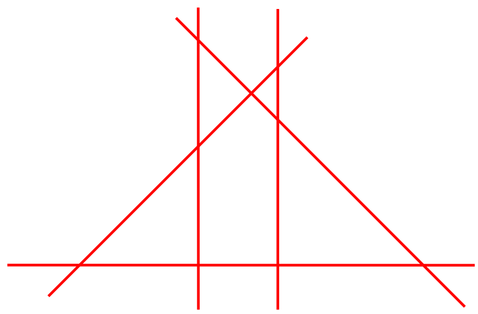
Answer：(D)

20. At most how many right triangles can be formed by five lines on the plane?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

【Solution】

Let the five lines are *a*, *b*, *c*, *d* and *e*. If two of them are perpendicular to each other, then the two lines and one of the other three lines can form a right triangle. Thus we can at most 3 right triangles. So if there are one pair or two pairs of perpendicular lines, then we can have at most 6 right triangles. If there are at least three pairs of perpendicular lines, then there exist two lines. Assume the two lines are *a* and *b*, perpendicular to the same line and hence the two lines, *a* and *b*, are parallel to each other. Thus *a* and any two of *c*, *d* and *e* can form a right triangle and hence we can get at most 3 right triangles. *b* and any two of *c*, *d* and *e* can form a right triangle and hence we can also get at most 3 right triangles. *c*, *d* and *e* can also form a right triangle, so there are at most $3+3+1=7$ right triangles which can be formed by five lines on the plane.



Answer：(D)

21. The International Article Number has 13 digits *ABCDEFGHIJKLM*. Here *M* is a check digit. Let $S=A+3B+C+3D+E+3F+G+3H+I+3J+K+3L$. If *S* is a multiple of 10, then *M* is chosen to be 0. Otherwise it is chosen to be $M=10-t$ where *t* is the remainder obtained when *S* is divided by 10. The Code for a certain Article Number is 6901020□09017. What is the missing digit?

6 901020□09017

【Solution】

From the conditions, we have

$$S=6+3\times9+0+3\times1+0+3\times2+0+3\times\square+0+3\times9+0+3\times1=72+3\times\square.$$

Since $M=7$, $10-7=3$ is the remainder obtained when $72+3\times\square$ is divided by 10. Thus the unit digit of $3\times\square$ is 1. So $\square=7$.

Answer：007

【Note】

The International Article Number is a code so that $A+3B+C+3D+E+3F+G+3H+I+3J+K+3L+M$ is divisible by 10.

22. What is the largest three-digit number which can be expressed as the sum of the cubes of three different positive integers?

【Solution】

Observe that $10^3=1000$, so we need to find three of $1^3=1$, $2^3=8$, $3^3=27$, $4^3=64$, $5^3=125$, $6^3=216$, $7^3=343$, $8^3=512$ and $9^3=729$ so that the sum of the three numbers is less than 1000 and as close to 1000 as possible.

If neither 8^3 nor 9^3 is one of the three numbers, then the maximum sum of the three numbers is $5^3+6^3+7^3=684$.

If 9^3 is one of the three numbers, then neither 8^3 nor 7^3 is not one of the three numbers. As 6^3 is one of the three numbers, we have $729+216=945$ and hence the largest number we can pick is 3^3 so that the sum is 973. As 6^3 is not one of the three numbers, the maximum sum of the three numbers is $4^3+5^3+9^3=918$.

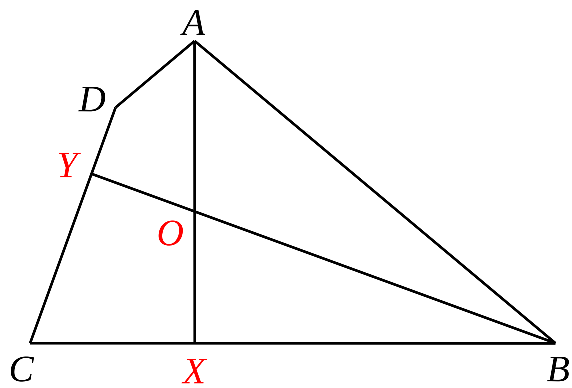
If 8^3 is one of the three numbers, then the maximum sum of the three numbers is $5^3+7^3+8^3=980$ since $6^3+7^3+8^3=1071>1000$.

So the largest three-digit number is 980.

Answer：980

23. The diagram shows a quadrilateral *ABCD* with $\angle CDA=150°$. The bisector of $\angle DAB$ is perpendicular to *BC* and the bisector of $\angle ABC$ is perpendicular to *CD*. What is the measure, in degrees, of $\angle BCD$?

【Solution 1】



Let the bisector of $\angle DAB$ intersect *BC* at *X*, the bisector of $\angle ABC$ intersect *CD* at *Y* and *AX* intersect *BY* at *O*. Since $\angle BCD+\angle XOY=180°$, $\angle BCD=\angle XOB=\frac{1}{2}(\angle DAB+\angle ABC)$, i.e., $\angle DAB+\angle ABC=2\angle BCD$. Since the sum of interior angles of a quadrilateral is $360°$, we have

$$\begin{aligned}360° &= \angle DAB+\angle ABC+\angle BCD+\angle CDA\\ &= 3\angle BCD+150°\end{aligned}$$

Thus $\angle BCD=70°$.

【Solution 2】

Assume $\angle ABC=2x°$. Since the bisector of $\angle ABC$ is perpendicular to *CD*, $\angle BCD=90°-x°$. And since the bisector of $\angle DAB$ is perpendicular to *BC*, $\angle DAB=2(90°-2x°)$. Now we have $360°=150°+2x°+90°-x°+2(90°-2x°)$

because the sum of interior angles of a quadrilateral is $360°$. Thus $3x° = 60°$ and hence $x = 20$. So $\angle BCD = 90° - 20° = 70°$.

Answer：070

24. Let a and b be positive real numbers such that $a^2 = b(b+1)$ and $b^2 = a+1$.

What is the value of $\frac{1}{a}+\frac{1}{b}$?

【Solution】

Since $b^2 = a+1$, we have $a = b^2 - 1$. We can also conclude that $a(b^2-1) = b(b+1)$ because $a^2 = b(b+1)$. Thus $a(b-1) = b$ and hence $ab = a+b$. Divide the equation by ab, we can get $\frac{1}{a}+\frac{1}{b} = \frac{a+b}{ab} = 1$.

Answer：001

25. Each blouse cost 40 dollars, each skirt 70 dollars and each pair of shoes 80 dollars. Fanny bought at least one item of each kind, and spent at most 800 dollars. A outfit consisted of one item of each kind, and two outfits were different if they differed in at least one item. At most how many different outfits could there be?

【Solution 1】

Suppose Fanny buys x blouses, y skirts and z pairs of shoes. Then we have $40x+70y+80z \le 800$ and want to find the maximum value of xyz. Divide the inequality by 40 to get $x+\frac{7}{4}y+2z \le 20$. Now we will discuss the possible value of y. Note that x, y and z are all positive integers, so $\frac{7}{4}y \le 17$, i.e., $y \le 9$.

As $y = 1$, we have $x+2z \le 18$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le 9$ and hence $xz \le \frac{9^2}{2} = 40\frac{1}{2}$. So the maximum value of xz is 40. Thus the maximum value of xyz is 40. It will occur when $x = 8$ and $z = 5$.

As $y = 2$, we have $x+2z \le 16$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le 8$ and hence $xz \le \frac{8^2}{2} = 32$. So the maximum value of xz is 32. Thus the maximum value of xyz is 64. It will occur when $x = 8$ and $z = 4$.

As $y = 3$, we have $x+2z \le 14$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le 7$ and hence $xz \le \frac{7^2}{2} = 24\frac{1}{2}$. So the maximum value of xz is 24. Thus the maximum value of xyz is 72. It will occur when $x = 6$ and $z = 4$.

As $y = 4$, we have $x+2z \le 13$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le \frac{13}{2}$ and hence $xz \le \frac{13^2}{8} = 21\frac{1}{8}$. So the maximum value of *xz* is 21. Thus the maximum value of *xyz* is 84. It will occur when $x = 7$ and $z = 3$.

As $y = 5$, we have $x + 2z \le 11$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le \frac{11}{2}$ and hence $xz \le \frac{11^2}{8} = 15\frac{1}{8}$. So the maximum value of *xz* is 15. Thus the maximum value of *xyz* is 75. It will occur when $x = 5$ and $z = 3$.

As $y = 6$, we have $x + 2z \le 9$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le \frac{9}{2}$ and hence $xz \le \frac{9^2}{8} = 10\frac{1}{8}$. So the maximum value of *xz* is 10. Thus the maximum value of *xyz* is 60. It will occur when $x = 5$ and $z = 2$.

As $y = 7$, we have $x + 2z \le 7$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le \frac{7}{2}$ and hence $xz \le \frac{7^2}{8} = 6\frac{1}{8}$. So the maximum value of *xz* is 6. Thus the maximum value of *xyz* is 42. It will occur when $x = 3$ and $z = 2$.

As $y = 8$, we have $x + 2z \le 6$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le 3$ and hence $xz \le \frac{3^2}{2} = 4\frac{1}{2}$. So the maximum value of *xz* is 4. Thus the maximum value of *xyz* is 32. It will occur when $x = 2$ and $z = 2$.

As $y = 9$, we have $x + 2z \le 4$. By the AM-GM inequality, $\sqrt{2xz} \le \frac{x+2z}{2} \le 2$ and hence $xz \le \frac{2^2}{2} = 2$. So the maximum value of *xz* is 2. Thus the maximum value of *xyz* is 18. It will occur when $x = 2$ and $z = 1$.

So there could be at most 84 different outfits when Fanny buys 7 blouses, 4 skirts and 3 pairs of shoes.

【Solution 2】

Suppose Fanny buys *x* blouses, *y* skirts and *z* pairs of shoes. Then we have $40x + 70y + 80z \le 800$ and want to find the maximum value of *xyz*. By the AM-GM inequality, $\sqrt[3]{40x \times 70y \times 80z} = \sqrt[3]{224000xyz} \le \frac{40x + 70y + 80z}{3} = \frac{800}{3}$. Cube the inequality and then get $224000\,xyz \le \frac{512000000}{27}$, i.e., $xyz \le \frac{512000000}{27 \times 224000} = 84\frac{127}{189}$.

So the maximum value of *xyz* is 84. It will occur when $x = 7$, $y = 4$ and $z = 3$.

Answer：084

【Note】The background is AM-GM inequality.