Solution Key to Second Round of IMAS 2017/2018
Upper Primary Division

1. Arrange 80 triangles in a row and color them black and white in a pattern as shown below. How many more black triangles than white triangles are there?

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(A) 10 (B) 16 (C) 18 (D) 20 (E) 25

【Solution】

From the picture, it shows that the coloring is periodic with period 5, consisting of 3 black triangles and 2 white ones. In each period, the number of black triangles is one more than the white ones. Since there are $80 \div 5 = 16$ periods, so there are 16 more black triangles.

Answer: (B)

2. The results of a math quiz of a certain class are as follows: 4 students got 100 points; the scores of 6 students are from 90 to 99; the scores of 18 students are from 80 to 89; while of 12 remaining students are from 70 to 79 and 10 students got below 69. The average of the class is 81.4. What is the total score of the class?

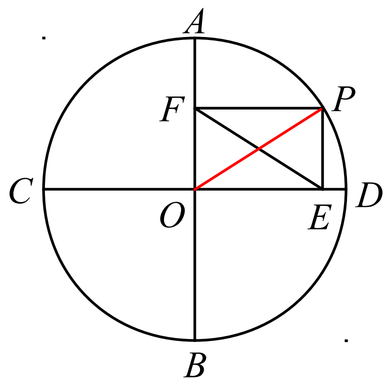
(A) 4050 (B) 3750 (C) 4070 (D) 3820 (E) Undetermined.

【Solution】

There are totally $4+6+18+12+10=50$ students in the class. So, the total score of the class is $81.4 \times 50 = 4070$.

Answer: (C)

3. Let AB and CD be two perpendicular diameters of a circle O. Draw two lines through any point P on the circle perpendicular to AB and CD, with intersections at F and E, respectively. If the diameter of the circle O is 8 cm, what is the length, in cm, of EF?



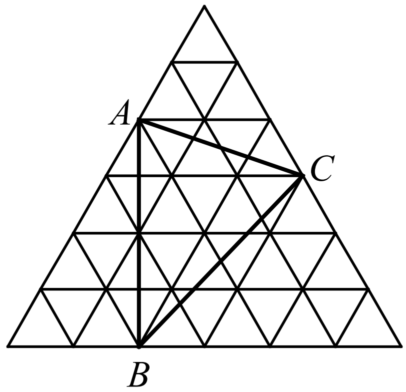
(A) 8 (B) 6 (C) 5 (D) 4 (E) 2

【Solution】

With the perpendicular conditions, it implies that $PEOF$ is a rectangle. Connect OP, then $OP = EF$. Since OP is radius of the circle, then $EF = 4$ cm.

Answer: (D)

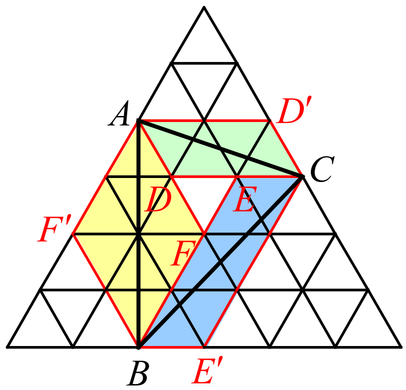
4. The figure below is composed of 36 small equilateral triangles, with each having an area of 1 cm^2. What is the area, in cm^2, of triangle ABC?



(A) 6 (B) 8 (C) 10 (D) 12 (E) 18

【Solution 1】

Mark points D, D', E, E', F and F' as in the figure. Then, the triangle ABC is divided into triangles ADC, BEC, AFB and DEF. Notice that:



(i) Triangle DEF has area 1 cm^2;

(ii) Triangle ADC has area half of parallelogram $ADCD'$, which is composed by 4 small equilateral triangles, thus the area of triangle ADC is $\frac{1}{2}\times 4=2\,\text{cm}^2$;

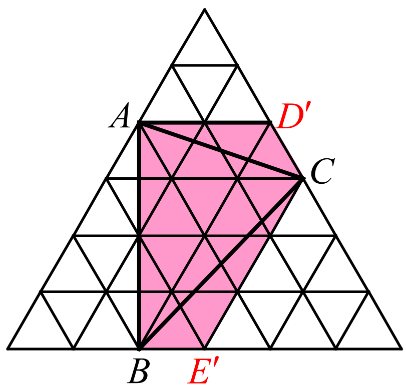
(iii) Similarly, triangle BEC has area half of parallelogram $BE'CE$, which is composed by 6 small equilateral triangles, thus the area of triangle BEC is $\frac{1}{2}\times 6=3\,\text{cm}^2$;

(iv) Triangle AFB has area half of parallelogram $AFBF'$, which is composed by 8 small equilateral triangles, thus the area of triangle AFB is $\frac{1}{2}\times 8=4\,\text{cm}^2$.

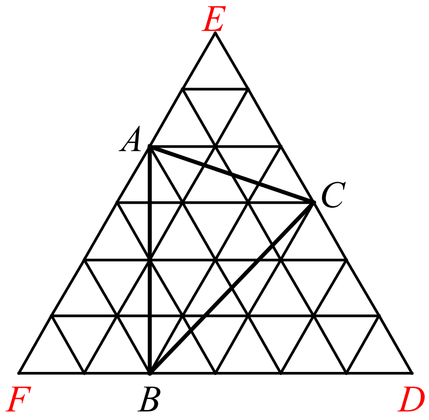
Triangle ABC has area $1+2+3+4=10$ cm^2.

【Solution 2】

Mark points D', E' as in the figure, then triangle ABC, ACD', BCE' compose to $ABE'CD'$, which consists 13 whole equilateral triangle and 4 half equilateral triangle and hence has area $13\times 1+4\times\frac{1}{2}=15$ cm^2. By【Solution 1】, triangle ACD' and BCE' have area 2 cm^2 and 3 cm^2, respectively. Then triangle ABC has area $15-2-3=10\,\text{cm}^2$.



【Solution 3】



Mark points D, E, F as in the figure, triangle DEF has area 36 cm^2. Notice that $AE=\frac{1}{3}EF$, $AF=\frac{2}{3}EF$, $FB=\frac{1}{3}FD$, $BD=\frac{2}{3}FD$, $CE=CD=\frac{1}{2}DE$.

Thus by the common angle theorem,

triangle ACE has area $\frac{1}{3}\times\frac{1}{2}\times 36=6$ cm^2,

triangle AFB has area $\frac{2}{3}\times\frac{1}{3}\times 36=8$ cm^2,

triangle ACE has area $\frac{2}{3}\times\frac{1}{2}\times 36=12$ cm^2.

So triangle ABC has area $36-6-8-12=10$ cm^2.

Answer: (C)

5. After removing the decimal part of a certain positive number, 5 times the sum of the integral part and the original positive number is 22.1. What is the value of this positive number?

（A）4.42　　（B）0.42　　（C）4.41　　（D）4　　（E）2.42

【Solution】

The sum of this number and its integer part is $22.1\div 5=4.42$, the non-integer part is then 0.42, the integer part is $4\div 2=2$, thus, the original number is 2.42.

Answer: (E)

6. A box contains identical balls where 7 are black, 5 are white and 8 are red balls. What is the least number of balls that must be taken out from the box to get balls of each color?

to make sure that we have at least one ball of each color?

【Solution】

To have at least one of the three colors, all the balls of the two colors of that have largest sizes must be taken out. That is $7+8=15$ balls. Taking one more ball ensures having balls at least one of the three colors, that is $15+1=16$.

Answer: 16

7. In the figure, $ABCD$ is a parallelogram, where E and F are midpoints of BC and CD, respectively. Now connect AE, AF, DE, BF, BD. The area of $ABCD$ is 4 cm^2. With three of A, B, C, D, E, F as vertices and present line segments as sides, how many triangles of area 1cm^2 can you find in the figure?

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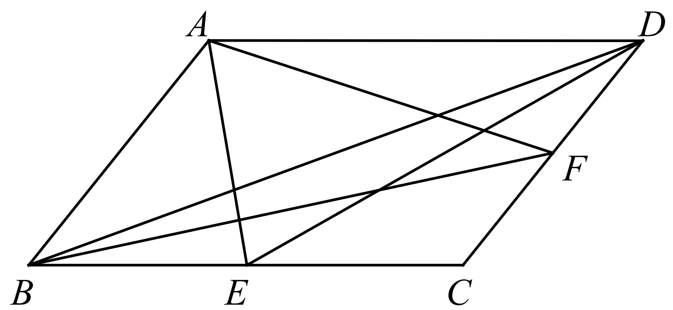


【Solution】

There are 10 triangles ABD, ABE, ABF, ADE, ADF, BCD, BCF, BDE, BDF, CDE in the figure. With the area of $ABCD$ being 4 cm^2, one knows:

(i) Each of triangles ABD, ABF, ADE, BCD has area $\frac{1}{2}\times 4 = 2\,cm^2$;

(ii) Since E is the midpoint of BC, each of triangles ABE, CDE, BDE has area $\frac{1}{2}\times\frac{1}{2}\times 4 = 1\,cm^2$;

(iii) Since F is the midpoint of CD, each of triangles ADF, BCF, BDF has area $\frac{1}{2}\times\frac{1}{2}\times 4 = 1\,cm^2$.

Totally there are 6 such triangles.

Answer: 6

8. A round table has 20 seats. Some seats are occupied such that a new person will always sit adjacent to someone wherever he is already seated. What is the least number of seats already occupied?

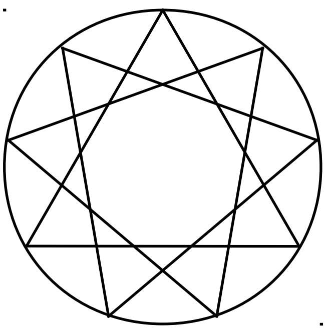
when a new person arrive, wherever he chooses to sit, he will always sit adjacent to someone.

【Solution】

A sequence of consecutive empty seats is at most of length 2. Thus every 3 consecutive seats is occupied at least once. Since $20 = 3\times 6 + 2$, there are at least $6 + 1 = 7$ seats occupied.

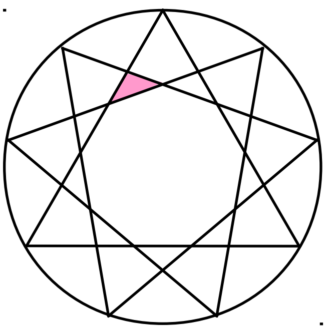
Answer: 7

9. Rotate an equilateral triangle inscribed in a circle 40 degrees clockwise and counter-clockwise, as shown in the figure below. How many triangles are there in the figure?

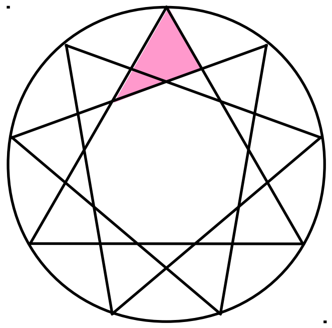


【Solution】

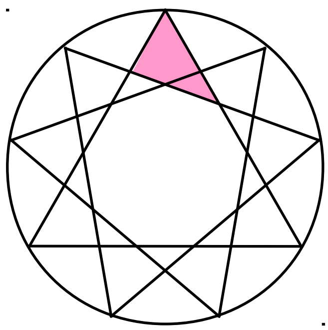
(i) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



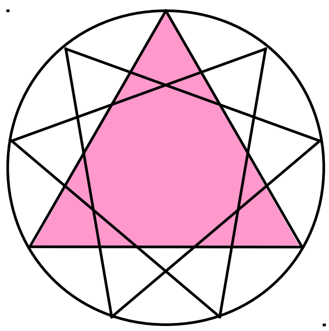
(ii) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



(iii) There are 9 triangles of same size but in different positions as the shaded triangle in the figure below.



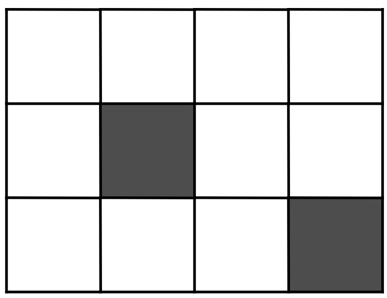
(iv) There are 3 triangles of same size but in different positions as the shaded triangle in the figure below.



Totally there are $9+9+9+3=30$ triangles.

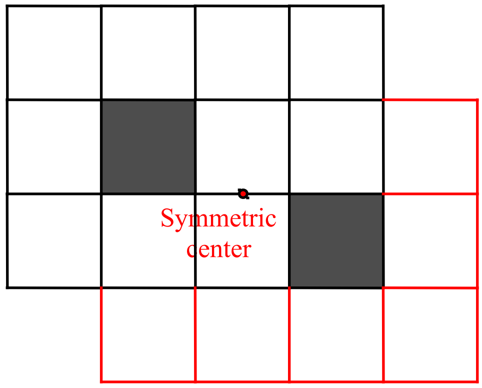
Answer: 30.

10. A rectangle is divided into 12 unit squares such that 10 are white and 2 are black, as shown in the figure below. To form a centrally symmetric picture by adding some white squares but no black squares, what is the least number of white squares needed?



【Solution】

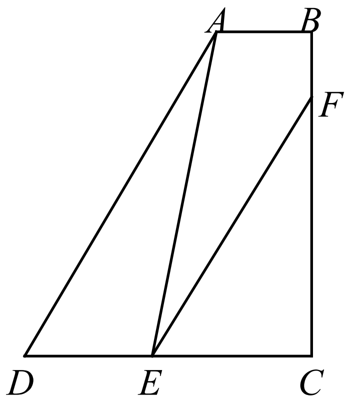
Since no more black unit squares are added, the symmetric center of the whole picture is the symmetric center of the two black unit squares. 4 white unit squares are already symmetric to one another with respect to this center. 6 more white unit squares are needed to be symmetric to those 6 alone white unit squares, as in the figure to the right.



Answer: 6

11. In the figure below, $ABCD$ is a right trapezoid where $\angle ABC = \angle BCD = 90°$, $AB = 3$ cm, $CD = 9$ cm. Points E and F are on CD and BC, respectively. If $BF = 2$ cm and AE with EF divides the trapezoid into three parts with equal area, what is the area, in cm^2, of $ABCD$?

Chú ý thêm dấu phẩy trước respectively. Dấu chấm và if sưa thành:
, such that BF = 2. If AE and EF



【Solution】

In the triangle ADE, the altitude on DE has the same length as BC, and the area of triangle ADE is one third of the area of $ABCD$. Thus $\dfrac{\frac{1}{2}DE \times BC}{\frac{1}{2}(AB + CD) \times BC} = \dfrac{1}{3}$, then $DE = \dfrac{1}{3}(AB + CD) = 4$ cm. Then $CE = CD - DE = 5$ cm. And the area of ADE is equal to the area of CEF implies $\dfrac{1}{2}DE \times BC = \dfrac{1}{2}CE \times CF$, plug in the lengths of DE, CE to get $CF = \dfrac{4}{5}BC$. Since $BF = BC - CF = \dfrac{1}{5}BC$ and $BF = 2$ cm, $BC = 10$ cm. Then the area of $ABCD$ is $\dfrac{1}{2} \times (AB + CD) \times BC = \dfrac{1}{2}(3 + 9) \times 10 = 60$ cm^2.

Answer: 60

12. A factory produces an order of parts. If the output per hour is 4 parts more than

the original speed, the time spent is $\frac{1}{10}$ less than the originally estimated time. If the speed is 6 parts less than the original speed per hour, the time spent is $\frac{1}{5}$ more than the original estimated time. How many parts does the factory originally produce per hour?

【Solution 1】

Suppose the original speed is v parts per hour and originally estimated time is t hours. Then

$$(v+4)(1-\frac{1}{10})t=(v-6)(1+\frac{1}{5})t$$

$$\frac{9}{10}(v+4)=\frac{6}{5}(v-6)$$

$$45(v+4)=60(v-6)$$

$$60v-45v=45\times 4+60\times 6$$

$$15v=540$$

$$v=36$$

Thus the original speed is 36 parts per hour.

【Solution 2】

The speed of 4 parts more than original and the speed of 6 parts less than original have difference 10 parts per hour, while the proportion of their time spent is $\frac{1-\frac{1}{10}}{1+\frac{1}{5}}=\frac{\frac{9}{10}}{\frac{6}{5}}=\frac{3}{4}$. Thus the faster speed produces $10\div(1-\frac{3}{4})=40$ parts per hour and the slower speed produces $40-10=30$ per hour and the original speed is $40-4=30+6=36$ parts per hour.

【Solution 3】

The proportion of their speed and the proportion of their time spent is inversely proportional to each other. Suppose the original speed is v parts per hour, then:

$$\frac{v}{v+4}=\frac{9}{10}$$

$$9v+36=10v$$

$$v=36$$

and

$$\frac{v}{v-6}=\frac{6}{5}$$

$$6v-36=5v$$

$$v=36$$

The original speed is 36 parts per hour. Answer: 36

13. A three-digit number is said to be "lucky" if it is divisible by 6 and by swapping its last two digits will give a number divisible by 6. How many "lucky" numbers are there?

【Solution】

Being divisible by 6 is equivalent to being divisible by both 2 and 3. Thus, the last two digits of a lucky number is even. A number is divisible by 3 if and only if the sum of all digits is divisible by 3. The remainder divided by 3 of the first number of the lucky number is determined by the last two digits. For any remainder, there are exactly 3 non-zero digits. Thus, the number of all lucky numbers is $5\times5\times3=75$.

Answer: 75

14. There is a sequence of five positive integers. Each number right after the first term is at least twice the number before it. If the sum of the five numbers is 2018, what is the least possible value of the last number?

Each term after the first term is at least twice the previous term.

【Solution】

Let x be the last number. Then the first four numbers are at most $\frac{x}{16}$, $\frac{x}{8}$, $\frac{x}{4}$ and $\frac{x}{2}$.

Thus $\frac{x}{16}+\frac{x}{8}+\frac{x}{4}+\frac{x}{2}+x\geq2018$,(5 marks) i.e. $x\geq\frac{2018\times16}{31}=1041\frac{17}{31}$, so x is at least 1042. (5 marks) Taking the five numbers as 65, 130, 260, 521 and 1042 satisfies the requirement of the problem. (5 marks) Hence the least possible value of the last number is 1042. (5 marks)

Answer: 1042

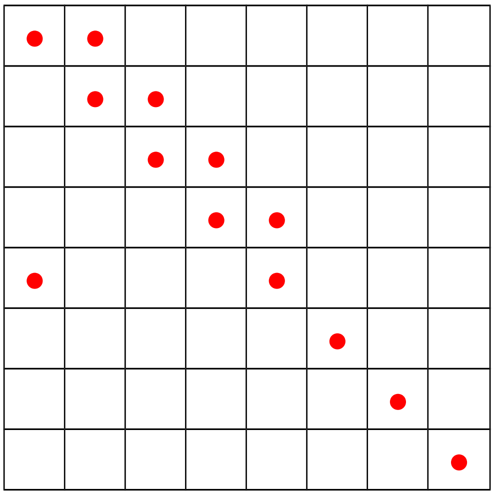
15. Some chess pieces are put on a 8×8 chess board, with at most 1 piece in each square. After taking all pieces on any chosen 4 rows and 4 columns, there is at least 1 piece left on the board. Find the least number of pieces originally on the board.

【Solution】Chú ý phần cho điểm: Chỉ cần chứng minh: 12 không thể được – 13 được – lấy được 1 VD là okie với bài tìm min (không cần chứng minh có cái 13 không được, không cần thiết).

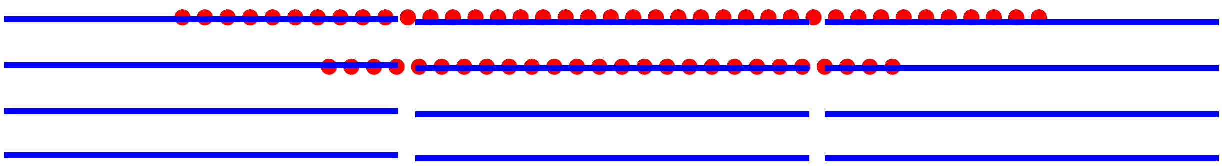
Suppose the total number of pieces is no more than 12. If there are at least 4 rows each with chesses more than 1, cover 4 such rows, the remaining pieces is at most $12-4\times2=4$, which can be covered by 4 columns, contradiction to that there is at least one piece left.

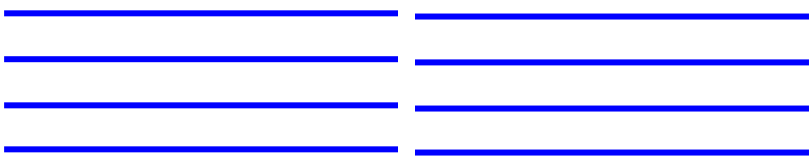
If there are less than 4 rows with pieces more than 1, covering 4 rows with the most pieces will leave all remaining row each contains no more than one piece. The remaining pieces can be covered by 4 columns, contradiction. (5 marks)

Put 13 pieces in the squares as the picture below will ensure that any 4 rows and 4 columns can not cover all pieces. (5 marks) Noticing that pieces in the upper left 5×5 square can not be covered by k rows and $5-k$ columns for any $k=1, 2, 3, 4$; pieces in the lower right 3×3 square can be covered only by totally 3 rows(columns). (5 marks)



It worth mentioning that some construction of 13 pieces arrangement is not correct, such as the examples below, each covered by 4 rows indicated by blue lines and 4 columns indicated by yellow area.





To summarize, the least number of pieces is 13. (5 marks)

Answer: 13